UNIT– 1

ALGEBRA

Learning objectives

- To understand and identify the basic features of law of indices, formulae of algebra, partial fractions, complex numbers, determinants & matrices, permutation & combination, binomial theorem in positive integral index.

1.1 LAW OF INDICES

Introduction: A power or an index is used to write product of numbers very compactly. The plural of index is indices. In this topic, we remind you how this is done and state a number of rules or laws, which can be used to simplify expressions involving indices.

Power or Indices : We write the expression $3 \times 3 \times 3 \times 3$ as 3^4.

We read this as "three to the power four or three raise to power four".

In the expression b^c, b is called the base and c is called the index.

Rules or Laws of Indices	Examples
First Rule $a^m \times a^n = a^{m+n}$	$2^5 \times 2^3 = 2^8$
(ii) $\frac{a^m}{a^n} = a^{m-n}$	$\frac{5^7}{5^3} = 5^{7-3} = 5^4$
(iii) $(a^m)^n = a^{mn}$	$(10^3)^7 = 10^{21}$
(iv) $a^0 = 1$	$10^0 = 1$
(v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{5}{6}\right)^2 = \frac{25}{36}$
(vi) $(ab)^n = a^n b^n$	$(2a)^5 = 2^5 a^5 = 32a^5$
(vii) $a^{-m} = \frac{1}{a^m}$	$(9)^{-2} = \frac{1}{9^2} = \frac{1}{81}$
(viii) $a^{\frac{n}{m}} = \sqrt[m]{a^n}$	$8^{2/3} = \sqrt[3]{8^2} = (8)^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$

(ix) $a^m = b^m \Rightarrow a = b$ $a^5 = b^5 \Rightarrow a = b$

(if the powers are equal then bases are equal)

(x) If $a^m = a^n \Rightarrow m = n$

(xi) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} = (ab)^{\frac{1}{n}}$

Examples:

(i) $a^2 \times a^5 = a^7$

(ii) $a^{-2} b^3 \times a^5 b^{-4} = a^3 b^{-1}$

(iii) $\left(\frac{3a^{-2}}{b^{-1}}\right)^3 = \frac{3^3 a^{-6}}{b^{-3}} = \frac{27b^3}{a^6}$

(iv) $\left(\frac{5a^2}{3b^3}\right)^2 \times \left(\frac{a}{b}\right)^{-2} = \frac{25a^4}{9b^6} \times \frac{a^{-2}}{b^{-2}} = \frac{25a^2}{9b^4}$

(v) $3^3 (x^3)^3 \times (y^4)^3 = 27x^9 y^{12}$

(vi) $\frac{a \times (ab^4)^2}{(a^2 \times b)^3} = \frac{a \times a^2 b^8}{a^6 b^3} = \frac{a^3 b^8}{a^6 b^3} = a^{-3} b^5$

(vii) $\sqrt{\frac{a^2}{b^6}} = (a^2 b^{-6})^{\frac{1}{2}} = (a^2)^{\frac{1}{2}} \times (b^{-6})^{\frac{1}{2}} = a \times b^{-3} = \frac{a}{b^3}$

EXERCISE - I

1. Simplify the following:

(i) $2^3 \times 2^4$

(ii) $8^{13} \div 8^5$

(iii) $(a^3)^4$

(iv) $3a^2 b^3 \times 4a^4 b^5$

(v) $\dfrac{a^2}{b^{-3}c^2}$

(vi) $\left(\dfrac{3a^{-2}}{b^{-1}}\right)^3$

(vii) $\left(\dfrac{5a^2}{3b^3}\right)^2 \times \left(\dfrac{a}{b}\right)^{-2}$

(viii) $\dfrac{a^3 \times a^4}{a^2}$

(ix) 7^{-2}

(x) $(3x^3\, y^4)^3$

ANSWERS

(i) 2^7 (ii) 8^8 (iii) a^{12} (iv) $12a^6\, b^8$ (v) $a^2\, b^3\, c^{-2}$ (vi) $\dfrac{27b^3}{a^6}$

(vii) $\dfrac{25a^2}{9b^4}$ (viii) a^5 (ix) $\dfrac{1}{49}$ (x) $27x^9\, y^{12}$

Formulae of Algebra

For any two numbers a and b

(1) $(a + b)^2 = a^2 + 2ab + b^2$ $\Rightarrow$ Square of a sum

(2) $(a - b)^2 = a^2 - 2ab + b^2$ $\Rightarrow$ Square of a difference

(3) $a^2 - b^2 = (a + b)(a - b)$ $\Rightarrow$ Difference of two squares

(4) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ $\Rightarrow$ Difference of two cubes

(5) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $\Rightarrow$ Sum of two cubes

(6) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ $\Rightarrow$ Cube of a sum

(7) $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$ $\Rightarrow$ Cube of a difference

Factorization Formula/Quadratic Formula

If a, b and c are real numbers, then $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: A quadratic equation can be solved by

(1) Using the concept of factorization

(2) Using the concept of quadratic formula

Example 1. Solve by factorization method; $x^2 + 7x + 10 = 0$.

Sol. In this method split the middle part 7 into two parts, such that their sum is +7 and product is +10. So numbers are 2 and 5.

$x^2 + (2x + 5x) + 10 = 0 \Rightarrow 1.x^2 + 7.x + 10 = 0$

$x^2 + 2x + 5x + 10 = 0 \Rightarrow x(x + 2) + 5(x + 2) = 0$

$(x + 2)(x + 5) = 0$

So either $x + 2 = 0$ or $x + 5 = 0$

If $x + 2 = 0 \Rightarrow x = -2$

If $x + 5 = 0 \Rightarrow x = -5$

Thus, –2, –5 are roots of given equation.

Example 2. Solve the quadratic equation by quadratic formula: $x^2 + 7x + 10 = 0$.

Sol. Comparing the given quadratic equation with $ax^2 + bx + c = 0$, we get

$a = 1,\ b = 7,\ c = 10$

Applying formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$, we get

$$\Rightarrow \quad x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{-7 \pm \sqrt{49-40}}{2} = \frac{-7 \pm \sqrt{9}}{2}$$

$$= \frac{-7 \pm 3}{2} = \frac{-7+3}{2}, \frac{-7-3}{2}$$

$$= \frac{-4}{2}, \frac{-10}{2} = -2, -5$$

Thus, –2, –5 are roots of given equation.

1.2 PARTIAL FRACTION

Fraction : An expression of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ is known as a fraction.

Polynomial : An expression of the type $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$, where a_0, a, …, a_n are constants, is called a polynomial.

For example: (a) $x^3 + 2x^2 + 7x + 2$, (b) $4x^4 + 7x^3 - 9x^2 + 3$

Degree of Polynomial : Degree of Polynomial is the power of highest term in x(variable). In example (a), degree is 3 and in example (b) degree is 4.

Polynomial with different degree's

Name	Degree	Exs
Constant	zero	7, 9, $\frac{11}{2}$ etc.
Linear	1	x + 1, x - 3, 5x - 3, etc.
Quadratic	2	$x^2 + 7x + 10$, $3x^2 + 7x - 11$ etc.
Cubic	3	$x^3 + 2x^2 + 7x + 2$, $x^3 - 1$ etc.

Rational Fraction : An expression of the type $\frac{p(x)}{q(x)}$, where p(x) and q(x) are polynomials and q(x) ≠ 0 is known as rational fraction.

For example; $\frac{2x+5}{x^2-5x+4}, \frac{1}{x^2-1}, \frac{x^2-5}{x^2-3x+2}$

A fraction is of two types :

(1) Proper Fraction : If degree of numerator is lower than degree of denominator, it is called proper fraction.

For example; $\frac{2x+5}{x^2-5x+4}, \frac{2x+1}{(2x-1)(x+2)}$ are proper fractions.

(2) Improper Fraction: If the degree of numerator a greater or equal to the degree of denominator, then it is called improper fraction.

For example; $\frac{x^3-5}{x^2-7x+12}, \frac{x^2-5}{x^2-3x+2}$ are improper fractions.

Partial Fraction: The simplest constituent fraction of a compound fraction is called its partial fraction and the process of separating a compound fraction into its simplest constituent fractions is called the resolution into partial fraction.

We have learnt this in previous class that

$$\frac{1}{x-2} + \frac{1}{x-3} = \frac{2x-5}{x^2-5x+6} \text{ (compound fraction)}$$

or $\frac{1}{x-2} + \frac{1}{x-3}$ are partial fractions of $\frac{2x-5}{x^2-5x+6}$

We shall now study how to perform the inverse process i.e. to decompose or break up a single fraction into a number of fractions having their denominator as the factor of denominator of original fraction.

Note: Improper fraction can be converted into proper by dividing numerator by denominator and written in the form :

Improper fraction = quotient + proper fraction *e.g.* $$\frac{x^3}{x^2-3x+2}=(x+3)+\frac{7x-6}{x^2-3x+2}$$

Now $\frac{7x-6}{x^2-3x+2}$ is a proper fraction and can be split into partial fractions.

Note : For resolving a improper fraction into partial fractions, first it should be converted into a proper fraction as explained above.

We have different types :

Type 1 : To resolve proper fraction into partial fraction with denominator containing non repeated linear factors only

For the proper fraction $\frac{p(x)}{q(x)}$, $q(x)\neq 0$ and degree of p(x) < degree of q(x)

The linear factors (ax + b), (cx + d) etc. split into addition with numerator A, B, etc.

For examples:

(i) $$\frac{1}{(ax+b)(cx+d)}=\frac{A}{ax+b}+\frac{B}{cx+d}$$

(ii) $$\frac{1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

(iii) $$\frac{1}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$$

where A, B & C are constants.

Example 3. Resolve $\frac{1}{(x+2)(x-3)}$ into partial fractions.

Sol. Given fraction is in proper form.

Consider $$\frac{1}{(x+2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-3}$$

Multiplying by the LCM (x + 2) (x – 3) on both sides, we get

$$1 = A(x - 3) + B(x + 2) \quad (1)$$

To find A, Put $x + 2 = 0 \Rightarrow x = -2$ in (1)

$$1 = A(-2-3) + B(0)$$

$$1 = A(-5) + 0 = -5A$$

$$A = -\frac{1}{5}$$

To find B, Put $x - 3 = 0 \Rightarrow x = 3$ in (1)

$$1 = A(0) + B(3 + 2)$$

$$1 = 0 + 5B = 5B$$

$$B = \frac{1}{5}$$

Substitute those value of A and B in (1)

$$\frac{1}{(x+2)(x-3)} = \frac{-\frac{1}{5}}{x+2} + \frac{\frac{1}{5}}{x-3}$$

$$= -\frac{1}{5(x+2)} + \frac{1}{5(x-3)}$$

Example 4. Resolve $\frac{2x-1}{x^2-8x+15}$ into partial fractions.

Sol. Since $x^2 - 8x + 15 = (x - 5)(x - 3)$

Consider $\frac{2x-1}{x^2-8x+15} = \frac{2x-1}{(x-5)(x-3)} = \frac{A}{x-5} + \frac{B}{x-3}$

Multiplying by the LCM $(x - 5)(x - 3)$ on both sides

$$2x - 1 = A(x - 3) + B(x - 5) \quad (1)$$

To find A, put $x - 5 = 0$ i.e. $x = 5$ in (1)

$$2(5) - 1 = A(5 - 3) + B(0)$$

$$9 = 2A + 0 = 2A \quad \Rightarrow \quad A = \frac{9}{2}$$

To find B, put $x - 3 = 0 \Rightarrow \quad x = 3$ in (1)

$$2(3) - 1 = A(0) + B(3 - 5)$$

$$5 = B(-2) = -2B$$

$$\Rightarrow \quad B = -\frac{5}{2}$$

Substituting value of A & B, the equation (1) become

$$\frac{2x-1}{x^2-8x+15} = \frac{\frac{9}{2}}{x-5} + \frac{-\frac{5}{2}}{x-3}$$

$$= \frac{9}{2(x-5)} - \frac{5}{2(x-3)}$$

Example 5. Resolve $\frac{x^3}{x^2-3x+2}$ into partial fractions.

Sol. Here given fraction is not in a proper fraction. Dividing x^3 by $x^2 - 3x + 2$, we get

$$\frac{x^3}{x^2-3x+2} = x + 3 + \frac{7x-6}{x^2-3x+2}$$

Now $\frac{7x-6}{x^2-3x+2}$ is in proper form and can be split into partial fractions.

Let $$\frac{7x-6}{x^2-3x+2} = \frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

i.e. $$7x - 6 = A(x - 2) + B(x - 1) \qquad (1)$$

To find A, put $x - 1 = 0$ i.e. $x = 1$ in (1), we get

$$1 = A(1 - 2) + B(0)$$

$$1 = -A \qquad \Rightarrow \qquad A = -1$$

To find B, Put $x - 2 = 0 \Rightarrow x = 2$ in (1), we get

$$7(2) - 6 = A(2 - 2) + B(2 - 1)$$

$$\Rightarrow \quad 14 - 6 = A(0) + B(1)$$

$$\Rightarrow \quad 8 = 0 + B$$

$$\Rightarrow \quad B = 8$$

Putting value of A and B in (A), we get

$$\frac{7x-6}{x^2-3x+2} = \frac{-1}{x-1} + \frac{8}{x-2}$$

$$\therefore \quad \frac{x^3}{x^2-3x+2} = x + 3 + \frac{-1}{x-1} + \frac{8}{x-2}$$

$$= x + 3 - \frac{1}{x-1} + \frac{8}{x-2}$$

Example 6. Resolve $\frac{x-4}{(x+4)(x^2-3x+2)}$ into partial fractions.

Sol. Since $x^2 - 3x + 2 = (x - 2)(x - 1)$

$$\therefore \quad \frac{x-4}{(x+4)(x^2-3x+2)} = \frac{x-4}{(x+4)(x-2)(x-1)}$$

Let

$$\frac{x-4}{(x+4)(x^2-3x+2)} = \frac{A}{x+4} + \frac{B}{x-2} + \frac{C}{x-1} \qquad (1)$$

Multiplying the LCM $(x + 4)(x - 2)(x - 1)$ on both side of (1)

$$x - 4 = A(x - 1)(x - 2) + B(x + 4)(x - 1) + C(x + 4)(x - 2) \qquad (2)$$

To find A, Put $x + 4 = 0 \Rightarrow x = -4$ in (2)

$-8 = A(-4 - 1)\,(-4 - 2) = A(-5)(-6)$

$-8 = A(30) \quad \Rightarrow \quad A = -\frac{8}{30} = -\frac{4}{15}$

$A = -\frac{4}{15}$

To find B, put $x - 2 = 0 \Rightarrow \quad x = 2$ in (2)

$-2 = B(2 + 4)\,(2 - 1)$

$-2 = B(6)(1) = 6B$

$B = -\frac{2}{6} = -\frac{1}{3} \quad \Rightarrow \quad B = -\frac{1}{3}$

To find C, put $x - 1 = 0 \Rightarrow x = 1$ in (2)

$-3 = C(1 + 4)(1 - 2) = C(5)(-1) = -5C$

$C = \frac{3}{5}$

Putting values of A, B & C in eqn**.** (1)

$$\frac{x-4}{(x+4)(x^2-3x+2)} = \frac{-4}{15(x+4)} - \frac{1}{3(x-2)} + \frac{3}{5(x-1)}$$

Type (ii): When the denominator contains repeated linear factors

Type (iii): When the denominator contains non-repeated quadratic factors

Type (iv): When the denominator contains repeated quadratic factors

EXERCISE - II

1. Resolve into the partial fractions :

(i) $\frac{1}{(x-3)(x-5)}$

(ii) $\frac{7x+1}{x^2-x-2}$

(iii) $\frac{5x-1}{(x-2)(x+1)}$

(iv) $\frac{x+1}{(x+3)(x^2-4)}$

(v) $\frac{2x-3}{(x-2)(x+3)}$

(vi) $\frac{1}{(1-x)(1-2x)(1-3x)}$

(vii) $\frac{5x-2}{x^2-2x-8}$

(viii) $\frac{x^3}{x^2-3x+2}$

(ix) $\frac{x^2}{(x-1)(x-2)(x-3)}$

(x) $\frac{x+5}{x^2+x}$

ANSWERS

(i) $\frac{-\frac{1}{2}}{x-3}+\frac{\frac{1}{2}}{x-5}$ (ii) $\frac{2}{x+1}+\frac{5}{x-2}$ (iii) $\frac{3}{x-2}+\frac{2}{x+1}$ (iv) $\frac{3}{20(x-2)}+\frac{1}{4(x+3)}=\frac{2}{5(x+2)}$

(v) $\frac{7}{5(x-2)}+\frac{7}{5(x+3)}$ (vi) $\frac{1}{2(1-x)}-\frac{4}{1-2x}+\frac{9}{2(1-3x)}$ (vii) $\frac{3}{x-4}+\frac{2}{x+2}$

(viii) $x+3-\frac{1}{x-1}+\frac{8}{x-2}$ (ix) $\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{9}{2(x-3)}$ (x) $\frac{5}{x}-\frac{4}{x+1}$

1.3 COMPLEX NUMBERS

Number System: We know the number system as

(1) Natural numbers, N = {1, 2, 3, …, }

(2) Whole Numbers, W = {0,1, 2, 3, …}

(3) Integers, Z = {…,–3, –2, –1, 0, 1, 2, 3, …}

(4) Rational numbers, Q = $\left\{\frac{p}{q}, p, q \in Z, q \neq 0\right\}$

(5) Irrational numbers – The numbers whose decimal representation is non–terminating and non repeating

e.g. $\sqrt{2}, \sqrt{3}$ etc.

(6) Real numbers (R) = (Rational numbers + Irrational numbers)

Let us take an quadratic equation; $x^2 + 7x + 12 = 0$ which has real root –4 and –3, *i.e.* solution of $x^2 + 7x + 12 = 0$ is $x = -4$ and $x = -3$ which are both real numbers.

But for the quadratic equation of the form $4x^2 - 4x + 5 = 0$, no real value of x satisfies the equation. For the solution of the equations of such types the idea of complex numbers is introduced.

Imaginary Numbers or Complex numbers:

Solution of quadratic equation $x^2 + 4 = 0$

or $x^2 = -4$

$$x = \sqrt{-4} = \sqrt{-1 \times 4} = \sqrt{-1} \times \sqrt{4} = i \times 2$$

where $i = \sqrt{-1}$ is an imaginary number called as iota.

The square root of a negative number is always imaginary number.

e.g. $\sqrt{-4}, \sqrt{-16}, \sqrt{-25}, \sqrt{-\frac{9}{16}}$ etc. are all imaginary numbers.

i.e. $\sqrt{-4} = 2i,\ \sqrt{-16} = 4i,\ \sqrt{-25} = 5i$

and $\sqrt{-\frac{9}{16}} = \frac{3}{4}i$

Thus the solution of quadratic equation $4x^2 - 4x + 5 = 0$ are :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(5)}}{2 \times 4}$$

$$x = \frac{4 \pm \sqrt{16 - 80}}{8}$$

$$x = \frac{4 \pm \sqrt{-64}}{8} = \frac{4 \pm 8i}{8} = \frac{4 + 8i}{8}, \frac{4 - 8i}{8}$$

$\Rightarrow$ $\frac{1 + 2i}{2}, \frac{1 - 2i}{2}$

Thus the given quadratic equation has complex roots.

Powers of iota (*i*)

(i) $i = \sqrt{-1}$

(ii) $i^2 = 1$

(iii) $i^3 = i^2.i = (-1)i = -i$

(iv) $i^4 = (i^2)^2 = (-1)^2 = 1$

(v) $i^5 = i^4.i = i$

(vi) $i^{24} = (i^4)^6 = (1)^6 = 1$

(vii) $i^{25} = (i^4)^6 i = 1 \times i = i$

(viii) $i^{4n+1} = (i^4)^n i = 1 \times i = i$

Real and Imaginary part of Complex number : A number of the form x + iy, where x and y are real numbers and i is an imaginary number with property $i^2 = -1$ i.e. $i = \sqrt{-1}$ is called a complex number. The complex number is denoted by Z.

$\therefore$ $Z = x + i y$

Here x is called real part of Z and denoted by Re(Z) and y is called imaginary part of Z, it is denoted by Im(Z).

Thus complex number Z = x + iy can be represented as

Z = real part + ***i*** (Imaginary part)

Examples of complex numbers are :

(i) Z = 2 + i, where Re(Z) = 2, Im(Z) = 1

(ii) Z = 4 – 7i, where Re(Z) = 4, Im(Z) = –7

(iii) Z = $-\frac{2}{3}+\frac{5}{3}i$, where Re(Z) = $-\frac{2}{3}$, Im(Z) = $\frac{5}{3}$

(iv) Z = 4 + 0i , where Re(Z) = 4, Im(Z) = 0

Here Im(Z) = 0, such type of complex number is known as purely real

(v) Z = 0 + 3i, where Re(Z) = 0, Im(Z) = 3

Hence Re(Z) = 0, hence given complex number is called purely imaginary number.

Properties of Complex Numbers:

(i) Equality of two complex number: Let $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$ are two complex numbers.

If $Z_1 = Z_2$ $\Rightarrow$ $x_1 + iy_1 = x_2 + iy_2$

Then $x_1 = x_2$ and $y_1 = y_2$

i.e. their real and imaginary part are separately equal.

(ii) if x + iy = 0, then x = 0 and y = 0 *i.e.* if a complex number is zero then its real part and imaginary part both are zero.

Example 7. Solve the equation 2x + (3x + y)i = 4 + 10i.

Sol. Using property (i)

2x = 4 and 3x + y = 10

$\Rightarrow$ x = 2, Putting value of x = 2 in

$3x + y = 10$, we get $\quad 6 + y = 10 \Rightarrow y = 10 - 6 = 4$

Hence x = 2, y = 4

Example 8. Find x & y if $\frac{1}{x} + \frac{1}{y}i = 2 + 3i$.

Sol. Equating real and imaginary part, we get

$$\frac{1}{x} = 2 \quad \Rightarrow \quad x = \frac{1}{2}$$

and $$\frac{1}{y} = 3 \quad \Rightarrow \quad y = \frac{1}{3}$$

Conjugate of a Complex Number

If $z = x + iy$ is a complex number than conjugate of Z is $x - iy$. It is denoted by $\bar{z}$.

$$\therefore \quad \bar{z} = x - iy$$

e.g. if z = 2 + 3i, then conjugate of z is $\bar{z} = 2 - 3i$.

Properties of conjugate of complex number

(i) $\overline{(\bar{z})} = z$

(ii) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 z_2} = \bar{z}_1.\bar{z}_2$

Example 9. Write conjugate of $z = 4i^3 + 3i^2 + 5i$.

Sol. Given that

$$z = 4i^3 + 3i^2 + 5i$$

$$= 4i^2 i + 3(-1) + 5i$$

$$z = -4i - 3 + 5i = -3 + i$$

Conjugate of z is $-3 - i$.

Sign of complex number in quadrant system (Fig. 1.1)

x–axis is called real axis
y–axis is called imaginary.

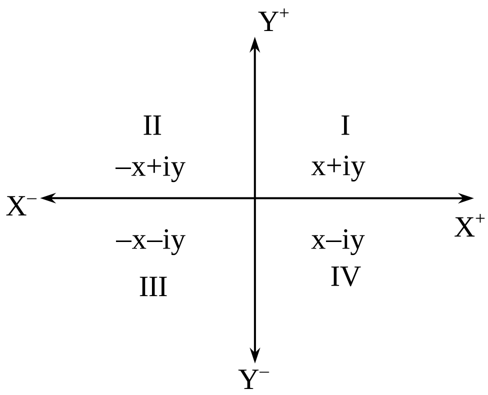

Figure .1.1

Polar and Cartesian form of a complex number.

Let z = x + iy is a complex number in Cartesian form represented by a Point P(x, y) in Argand plane (XY–plane) as shown in Fig. 1.2

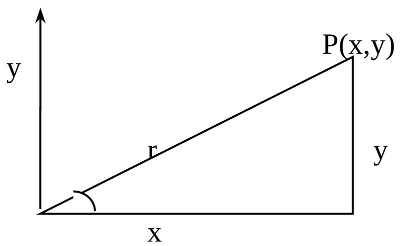

Figure .1.2

Then $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$

$\therefore$ $y = r\sin\theta$ …(1) and $x = r\cos\theta$ (2)

The complex number $Z = x + iy = r\cos\theta + i\,r\sin\theta = r[\cos\theta + i\sin\theta]$ is a polar form of given complex number. Squaring and adding (1) and (2), we get

$$x^2 + y^2 = r^2[\cos^2\theta + \sin^2\theta]$$

$$= r^2[1]$$

$\Rightarrow$ $r = \sqrt{x^2 + y^2}$ $\Rightarrow$ Modulus of z = x + iy *i.e.* r is called the modulus of z

Dividing (1) by (2), we get

$\tan\theta = \dfrac{y}{x}$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) \Rightarrow$ The argument or amplitude of z. *i.e.* θ is called the argument or amplitude of z.

$\therefore$ Polar form of a complex number z is

$$Z = x + iy = r[\cos\theta + i\sin\theta]$$

Thus $Z = x + iy$ is Cartesian form of Z.

and $Z = r[\cos\theta + i\sin\theta]$ is Polar form of Z

Where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

Conversion from Cartesian Form to Polar Form

Let x + iy be a complex number in Cartesian form, then we have to convert into polar form

$$x + iy \xrightarrow{\text{Change}} r(\cos\theta + i\sin\theta)$$

Putting the value of r and θ, we get required form

i.e. $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$\therefore$ $x + iy = r(\cos\theta + i\sin\theta)$

$$= \sqrt{x^2 + y^2}\left[\cos(\tan^{-1}\left(\frac{y}{x}\right)\right] + i\sin\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Example 10. Express $1 + \sqrt{3}i$ into polar form.

Sol. Let z = $1 + \sqrt{3}i$, here x = 1, $y = \sqrt{3}$

Polar form of complex number is

$$z = r[\cos\theta + i\sin\theta]$$

We know, $r = \sqrt{x^2 + y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3} = \tan\frac{\pi}{3}$$

$\Rightarrow$ $\theta = \frac{\pi}{3}$

$\therefore$ Required polar form is

$$z = r[\cos\theta + i\sin\theta] = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

Example 11. Convert 1 – i into polar form.

Sol. Let Z = 1 – i, then x = 1, y = –1

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1, \text{ z lies in IV quadrant}$$

$$\tan\theta = -1 = \tan\left(2\pi - \frac{\pi}{4}\right) = \tan\frac{7\pi}{4}$$

$$\theta = \frac{7\pi}{4}$$

$\therefore$ Required form of Z is

$$Z = r(\cos\theta + i\sin\theta)$$

$$Z = \sqrt{2}[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}]$$

Conversion from Polar form to Cartesian Form

Let Z = r(cos θ + i sin θ) be the polar form and x + iy be its rectangular form.

Put x = r cos θ, y = r sin θ to get required form.

Example 12. Convert 4(cos 300° + i sin 300°) into Cartesian from.

Sol. 4(cos 300° + i sin 300°) $\xrightarrow{\text{change}}$ x + iy

Put $x = 4\cos 300^o = 4\cos(360^o - 60^o) = 4\cos 60^o = 4 \times \frac{1}{2} = 2$

$y = 4\sin 300° = 4\sin(360° - 60°) = -4\sin 60° = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$

∴ Required Cartesian form is

$$x + iy = 2 - 2\sqrt{3}i$$

Modulus and Amplitude of a Complex Number

If z = x + iy is a complex number, then

a) $r = \sqrt{x^2 + y^2} = \sqrt{(\text{real part})^2 + (\text{Imaginary})^2}$ is known as Modulus of z.

it is denotes by | z |.

Thus Modulus of z = | z | = r = $\sqrt{x^2 + y^2}$

(b) Amplitude of z

$\tan\theta = \frac{y}{x}$ i.e. $\tan\theta = \frac{-\text{Im}(z)}{\text{Re}(z)}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is known as Amplitude or argument of z.

Example 13. Find the Modulus and Amplitude of a + ib.

Sol. Here z = a + ib

(i) Modulus, |z| = $\sqrt{a^2 + b^2}$

(ii) Amplitude, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Example 14. Find Modulus and amplitude of the complex number 1 + i.

Sol. Here z = 1 + I i.e. x = 1, y = 1

(i) Modulus | z | = $\sqrt{x^2 + y^2} = \sqrt{1 + 1} = \sqrt{2}$

(ii) Amplitude : $\tan\theta = \frac{y}{x} = \frac{1}{1}$

$$\tan\theta = 1 = \tan\frac{\pi}{4}$$

$$\theta = \frac{\pi}{4}$$

Example 15. Find the Modulus and Amplitude of the complex number –1 + i.

Sol. Let z = –1 + i

Compare it with z = x + iy, we get

x = –1, y = 1

(1) Modulus of z = | z | = $\sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

(2) Amplitude of z $\Rightarrow \tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$

Complex number –1 +i lie in II quadrant.

θ also lie in IInd quadrant.

$$\tan\theta = -1 = \tan(180° - 45°) = \tan 135°$$

$\therefore$ $\theta = 135°$ or $\theta = \frac{3\pi}{4}$.

Example 16. Find modulus of each of the complex numbers 6 + 7i and 1 + 10i.

Sol. Let z_1 = 6 + 7i, z_2 = 1 + 10i

Then Modulus of $z_1 = |z_1| = \sqrt{6^2 + 7^2} = \sqrt{36 + 49} = \sqrt{85}$

Modulus of z_2 = | z_2 | = $\sqrt{(1)^2+(10)^2} = \sqrt{1+100} = \sqrt{101}$

Example 17. Find modulus and amplitude of the complex number $-1+\sqrt{3}i$.

Solution : Let z = $=1+\sqrt{3}\,i$

then $\quad x = -1,\; y = \sqrt{3}$

(1) Modulus of z = | z | = $\sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = 2$

(2) Amplitude of z $\Rightarrow$ $\tan\theta = \dfrac{y}{x} = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$

Complex number lie in IInd quadrant.

$\therefore \quad \tan\theta = -\sqrt{3} = \tan\left(\pi-\dfrac{\pi}{6}\right) = \tan\dfrac{5\pi}{6}$.

$\theta = \dfrac{5\pi}{6}$.

Addition, Subtraction, Multiplication and Division of Complex Numbers :

(i) Addition of Complex Numbers : Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

Then $\quad z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$= (x_1 + x_2) + i(y_1 + y_2)$

i.e add real parts and Im. parts separately.

Note : Addition of two complex numbers is also a complex number.

Example 18. Let $z_1 = 7 + 3i$, $\quad z_2 = 9 - i$ then

$z_1 + z_2 = (7 + 3i) + (9 - i)$

$= (7 + 9) + i(3 - 1) = 16 + 2i$

Note $\quad z_1 + z_2 = z_2 + z_1$

(ii) Subtraction of Complex Numbers :

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex number.

then $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$

$= (x_1 - x_2) + i(y_1 - y_2)$

Note – Difference of two complex numbers is also complex number.

Example 19. Let $z_1 = 2 + 4i$, $z_2 = 7 + 5i$

then $z_1 - z_2 = (2 + 4i) - (7 + 5i)$

$= (2 - 7) + i(4 - 5)$

$= -5 - i$

Note $z_1 - z_2 \neq z_2 - z_1$

(iii) Multiplication of Complex Numbers

(a) In Cartesian form

Let $z_1 = x + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$ $(i^2 = -1)$

$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

(b) In polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_2)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex number in polar form.

Then $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

Note : Multiplication of two complex number is also a complex number.

(iv) Division of two complex numbers

(a) In Cartesian form

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{x_1 x_3 - ix_1 y_2 + ix_2 y_1 - iy_1 y_2}{x_2^2 - i^2 y_2^2}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2} \qquad (i^2 = -1)$$

$$\therefore \quad \frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$$

Note : Division of two complex numbers is also a complex number.

(b) In Polar form

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + \sin\theta_1)}{r_2(\cos\theta_2 + \sin\theta_2)}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Example 20. if $z_1 = -2 + 4i$ and $z_2 = 1 - 3i$, then find $z_1 z_2$.

Sol. $z_1 z_2 = (-2 + 4i)(1 - 3i)$

$= -2 + 6i + 4i - 12i^2 \qquad (i^2 = -1)$

$= -2 + 10i + 12$

$= 10 + 10i$

Example 21. if $z_1 = 5(\cos 30° + i \sin 30°)$ and $z_2 = 2(\cos 30° + i \sin 30°)$. Find $z_1 z_2$.

Sol. $z_1 z_2 = 5 \times 2[\cos(30 + 30) + i\sin(30 + 30)]$

$= 10[\cos 60° + i \sin 60°]$

$$= 10\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$= 5 + 5\sqrt{3}i$

Example 22. If $z_1 = 2 - i$, $z_2 = 2 + i$, then

$$\frac{z_1}{z_2} = \frac{2-i}{2+i} = \frac{2-i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{(2-i)^2}{(2)^2 - (i)^2} = \frac{4+i^2-4i}{4+1}$$

$$= \frac{3-4i}{5} = \frac{3}{5} - \frac{4i}{5}$$

Example 23. If $z_1 = 5 + 7i$, $z_2 = 9 - 3i$, find $\frac{z_1}{z_2}$

Sol. $$\frac{z_1}{z_2} = \frac{5+7i}{9-3i} = \frac{5+7i}{9-3i} \times \frac{9+3i}{9+3i}$$

$$= \frac{45+15i+63i+21i^2}{(9)^2-(3i)^2}$$

$$= \frac{(45-21)+i(15+63)}{81+9} = \frac{24+78i}{90}$$

$$= \frac{24}{90} + \frac{78}{90}i$$

Example 24. If $z_1 = 50[\cos 50° + i \sin 50°]$ and $z_2 = 10[\cos(-10°) + i \sin(-10°)]$

then $$\frac{z_1}{z_2} = \frac{50}{10}[\cos(50-(-10)+i\sin[50-(-10)]$$

$$= 5[\cos 60° + i \sin 60°]$$

$$= 5\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$$

$$= \frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

Example 25. Find modulus of $z = 4i^3 + 3i^2 + 5i$

given $z = 4(-i) - 3 + 5i$

$z = -4i - 3 + 5i$

$z = -3 + i$

$|z| = \sqrt{(-3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$

(V) Multiplicative Inverse of a Complex Number

Let z = x + iy be a complex number then multiplicative inverse of z is $\frac{1}{z}$

i.e. $$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Example 26. Find the multiplicative inverse (MI) of 1 – 2i.

Sol. MI of z is given by $\frac{1}{z} = \frac{1}{1-2i}$

$$= \frac{1}{1-2i} \times \frac{1+2i}{1+2i} = \frac{1+2i}{1+4}$$

$$= \frac{1}{5} + \frac{2}{5}i$$

Example 27. Find the value of x and y if 3x + (2x – y)i = 6 – 3i.

Sol. Equating real and imaginary part, we get

$3x = 6 \quad \Rightarrow \quad x = 2$

and $2x - y = -3$

$2(2) - y = -3 \quad \Rightarrow \quad 4 - y = -3$

$-y = -7 \Rightarrow \quad y = 7$

Example 28. Express in complex form $\frac{2-i}{(1-2i)^2}$.

Sol. $$\frac{2-i}{(1-2i)^2} = \frac{2-i}{1+4i^2-4i} = \frac{2-i}{-3-4i}$$

$$= \frac{2-i}{-3-4i} \times \frac{-3+4i}{-3+4i} = \frac{-6+8i+3i-4i^2}{9+16}$$

$$= \frac{-2+11i}{25} = \frac{-2}{25} + \frac{11}{25}i$$ which is required x + iy form

Example 29. Simplify $\left(\frac{1}{2}+2i\right)^3$

Sol. $$\left(\frac{1}{2}+2i\right)^3 = \left(\frac{1}{2}\right)^3 + (2i)^3 + 3\left(\frac{1}{2}\right)(2i)\left[\frac{1}{2}+2i\right]$$

$$= \frac{1}{8} + 8i^3 + 3i(\frac{1}{2}+2i)$$

$$= \frac{1}{8} - 8i + \frac{3}{2}i + 6i^2$$

$$= \frac{1}{8} - 8i + \frac{3}{2}i - 6 = \frac{-47}{8} - \frac{13}{2}i$$

Example. 30. Find multiplicative inverse of z = $(6 + 5i)^2$.

Sol. $z = (6+5i)^2 = 36 + 25i^2 + 60i$

$= 36 - 25 + 60i$

$= 11 + 60i$

M.I. of $$z = \frac{1}{z} = \frac{1}{11+60i} = \frac{1}{11+60i} \times \frac{11-60i}{11-60i}$$

$$= \frac{11-60i}{121+3600} = \frac{11-60i}{3721}$$

$$= \frac{11}{3721} - \frac{60}{3721}i$$

Example 31. Express $\frac{1}{3+i} - \frac{1}{3-i}$ in x + iy form

Sol. $$\frac{1}{3+i} - \frac{1}{3-i} = \frac{3-i-3-i}{(3+i)(3-i)} = \frac{-2i}{9+1}$$

$= \dfrac{0-2i}{10}$ $\quad = 0-\dfrac{1}{5}i$ is the required form.

Example 32. Express $\dfrac{(3+i)(4-i)}{5+i}$ in the form a + ib.

Sol. $\dfrac{(3+i)(4-i)}{5+i} = \dfrac{12-3i+4i-i^2}{5+i} = \dfrac{13+i}{5+i}$

$\dfrac{13+i}{5+i} \times \dfrac{5-i}{5-i} = \dfrac{65-13i+5i-i^2}{25+1} = \dfrac{66-8i}{26}$

$\dfrac{66}{26}-\dfrac{8}{26}i$ $\quad = \dfrac{33}{13}-\dfrac{4}{13}i$ is required a + ib form.

Example 33. Find Modulus and amplitude of $\dfrac{1+i}{1-i}$.

Sol. Let $z = \dfrac{1+i}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{(1+i)^2}{1^2+i^2} = \dfrac{(1+i)^2}{1+1}$

$z = \dfrac{1+i^2+2i}{2} = \dfrac{1-1+2i}{2} = \dfrac{2i}{2} = i$

z = i = 0 + i

(a) Modulus $|z| = \sqrt{(0)^2+(1)^2} = 1$

(b) Amplitude $= \tan\theta = \dfrac{y}{x} = \dfrac{1}{0} = \infty = \tan\dfrac{\pi}{2}$

$\theta = \dfrac{\pi}{2}$.

Example 34. Simplify $1+i^{100}+i^{10}+i^{50}$.

Sol. $1+i^{100}+i^{10}+i^{50} = 1+(i^4)^{25}+i^8.i^2+i^{48}.i^2$

$= 1+(1)^{25}+(i^4)^2 i^2+(i^4)^{12} i^2$

$=1 + 1 + (1).i^2 + (1)\ i^2$

$= 1 + 1 - 1 - 1 = 0$

EXERCISE –III

Questions on complex numbers:

1. If (x + iy) (2 – 3i) = 4 – I, find x and y.

2. If (a – 2bi) + (b – 3ai), find a and b.

3. Find real value of x and y if (1 – i)x + (1 + i)y = 1 – 3i.

4. Evaluate (i) i^{25} (ii) i^{19}

5. Find modulus and Amplitude of following

   (i) $1+\sqrt{3}i$

   (ii) 1 + i

   (iii) $4\sqrt{3}+4i$

   (iv) $z = \frac{3+2i}{4-5i}$

6. Add 2 + 3i and 5 – 6i

7. Subtract 7 – 5i from 2+4*i*

8. Simplify (5 + 5i) (4 – 3i)

9. If $z_1 = 1 + 3i$, $z_2 = 2 + i$, find z_1z_2

10. Write $\frac{3+4i}{2-3i}$ in x + iy form

11. Simplify $\frac{4-7i}{3-2i}$

12. Find multiplicative inverse of 3 + 4i

13. Write conjugate of –3 + 2i

14. Write conjugate of $\frac{3+2i}{4-5i}$

15. Express $\frac{(2+3i)^2}{1-i}$ is x + iy form

16. $z_1 = 2(\cos 60° + i \sin 60°)$, $z_2 = 4(\cos 15° + i \sin 15°)$, find $\frac{z_1}{z_2}$.

17. Write $\left(\frac{3}{2}+\frac{\sqrt{5}}{2}i\right)^2$ in x + iy form

18. Find modulus and conjugate of $\frac{2+7i}{3+2i}$

19. Express $\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}$ into x + iy form

20. Express $\frac{2+4i}{2-3i}$ in complex form x + *i*y

21. Write $\frac{(1-i)(2-i)(3-i)}{1+i}$ in x + iy form

22. Find the conjugate of $(3 - 7i)^2$

23. Find the amplitude of z if z = $\frac{-1-\sqrt{3}i}{2}$

24. The value of $\frac{i+i^2+i^3+i^4+i^5}{i+1}$ is

(a) $\frac{1-i}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) $\frac{i+1}{2}$

25. If $\left(x+\sqrt{y}\right)\left(p+\sqrt{q}\right) = x^2 + y^2$ then

(a) p = x, q = y (b) $p = x^2$, $q = y^2$ (c) p = y, q = x (d) None of these

26. $(3a^{-2} + 2b^{-1})^2 =$

(a) $9a^{-4} + 4b^{-2} + 12a^{-2}b^{-1}$ (b) $9a^0 + 4b^0 + 12a^{-2}b^{-1}$

(c) $9a^{-2} + 4b^{-4} + 12a^{-2}b^{-1}$ (d) None of these

27. The value of $\sqrt{(-1)\,(-1)}$ is

(a) 1 (b) -1 (c) *i* (d) -*i*

28. The roots of the equation, $ax^2 + 2bx + c = 0$ are

(a) $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ (b) $\frac{-b\pm\sqrt{4b^2-4ac}}{2a}$

(c) $\frac{-2b\pm\sqrt{4b^2-4ac}}{2a}$ (d) None of these

29. The factorization of $x^2 - 5x + 6 = 0$ is

(a) (x+2) (x-3) (b) (x+2) (x+3)

(c) (x-2) (x-3) (d) (x-2) (x+3)

30. In a proper fraction;

(a) The degree of numeration is equal to the degree of denominator

(b) The degree of numeration is less than the degree of denominator

(c) The degree of numeration is more than the degree of denominator

(d) None of these

31. The partial fractions of $\frac{x+3}{x^2+x}$ are

(a) $\frac{2}{x} - \frac{3}{x+1}$ (b) $-\frac{2}{x} + \frac{3}{x+1}$

(c) $\frac{3}{x} - \frac{2}{x+1}$ (d) $\frac{3}{x} + \frac{2}{x+1}$

32. The multiplicative inverse of 1+*i* is

(a) $\frac{1}{2}(1-i)$ (b) $\frac{1}{2}(1+i)$

(c) (1 - *i*) (d) None of these

33. The real & imaginary parts of $\frac{(2+3i)^2}{1-i}$ are

(a) $\frac{22}{5}$ & $\frac{19}{5}$ (b) $\frac{-22}{5}$ & $\frac{-19}{5}$ 0028c) $\frac{-22}{5}$ & $\frac{19}{5}$ (d) None of these

34. Amplitude of $\frac{-1-\sqrt{3}\,i}{2}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

35. Conjugate of $(2 + i)^2$ is

(a) -3 + 4*i* (b) 3 - 4*i* (c) -3 - 4*i* (d) None of these

ANSWERS

1. $x = \frac{11}{3},\ y = \frac{10}{3}$ 2. $a = -12,\ b = 17$ 3. $x = 0,\ y = -1$ 4. (i) i (ii) $-i$

5. (i) $r = 2,\ \theta = \frac{\pi}{3}$ (ii) $r = \sqrt{2},\ \theta = \frac{\pi}{4}$ (iii) $r = 8,\ \theta = \frac{\pi}{3}$ (iv) $|z| = \frac{\sqrt{533}}{41}$, $\boldsymbol{\theta} = \tan^{-1}\left(\frac{23}{2}\right)$

6. $7 - 3i$ 7. $-5 + 9i$ 8. $[35 + 5i]$ 9. $-1 + 7i$ 10. $\frac{-6}{13} + \frac{17}{13}i$ 11. $10\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

12. $\left(\frac{3}{25}\right) + \left(\frac{-4}{25}i\right)$ 13. $-3 - 2i$ 14. $\frac{2}{41} - \frac{23}{41}i$ 15. $\frac{-22}{5} + \frac{19}{5}i$ 16. $-\frac{1}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$

17. $1 + \frac{3\sqrt{5}}{2}i$ 18. $|z| = 1,\ \bar{z} = \frac{12}{13} - \frac{5}{13}i$ 19. $\frac{-8}{29}$ 20. $-\frac{6}{13} + \frac{17}{13}i$ 21. $5 + 5i$

22. $-40 - 42i$ 23. $\frac{\pi}{3}$ 24. (d) 25. (d) 26. (a) 27. (b)

28. (c) 29. (c) 30. (b) 31. (c) 32. (a) 33. (c)

34. (d) 35. (b)

1.4 LOGARITHM

Definition: If y and a are positive real numbers$(a \neq 1)$, then $x = log_a y$ if and only if $a^x = y$.

The notation $log_a y$ is read as "log to the base a of y ". In the equation $x = log_a y$, x is known as the **logarithm**, a is the **base** and y is the **argument**.

Note: 1. The above definition indicates that a logarithm is an exponent.

Logarithm and Exponent

$x = log_a y$ if and only if $a^x = y$

Base

2. Logarithm of a number may be negative but the argument of logarithm must be positive. The base must also be positive and not equal to 1.

3.

Logarithmic Form	Exponential Form
$x = log_a y$	$a^x = y$

4. Logarithm of zero doesn't exist.

5. Logarithms of negative real numbers are not defined in the system of real numbers.

6. Log to the base "10" is called Common Logarithm and Log to the base "e" is called Natural Logarithm.($e = 2.7182818284 \ldots$)

7. If base of logarithm is not given, it is considered to be Natural Logarithm.

Some Examples of logarithmic form and their corresponding exponential form:

S. No.	Logarithmic form	Exponential form
1	$5 = log_2 32$	$2^5 = 32$
2	$4 = log_3 81$	$3^4 = 81$
3	$3 = log_5 125$	$5^3 = 125$
4	$4 = log_{10} 10000$	$10^4 = 10000$
5	$-2 = log_7 \left(\frac{1}{49}\right)$	$7^{-2} = \frac{1}{49}$
6	$0 = log_e 1$	$e^0 = 1$

Why do we study logarithms: Sometimes multiplication, subtraction and exponentiation become so lengthy and tedious to solve. Logarithms covert the problems of multiplication into addition, division into subtraction and exponentiation into multiplication, which are easy to solve.

Properties of Logarithms: If a, b and c are positive real numbers, $a \neq 1$ and n is any real number, then

1. **Product property:** $\boldsymbol{log_a(b.c) = log_a b + log_a c}$

   For ex: $log_{10}(187) = log_{10}(11 \times 17) = log_{10} 11 + log_{10} 17$

2. **Quotient property:** $\boldsymbol{log_a \left(\frac{b}{c}\right) = log_a b - log_a c}$

   For ex: $log_7 \left(\frac{51}{7}\right) = log_7 51 - log_7 7$

3. **Power property:** $\boldsymbol{log_a b^n = n.log_a b}$

For ex: $log_{10}(10000) = log_{10}(10^4) = 4.log_{10}10$

4. **One to One property: $\boldsymbol{log_a b = log_a c}$ if and only if $\boldsymbol{b = c}$.**

   For ex: If $log_{10}(a) = log_{10}(15)$ then $a = 15$.

5. $\boldsymbol{log_a 1 = 0}$

   For ex: $log_{10}(1) = 0$, $log_2(1) = 0$, $log_e(1) = 0$ etc.

6. $\boldsymbol{log_a a = 1}$

   For ex: $log_{10}(10) = 1$, $log_e(e) = 1$ etc.

7. $\boldsymbol{log_a a^n = n}$

   For ex: $log_{10}10^4 = 4$

8. $\boldsymbol{a^{log_a(n)} = n}$**, where** $\boldsymbol{n} > 0$

   For ex: $2^{log_2(8)} = 8$

9. **Change of base property:** $\boldsymbol{log_a\, b = \frac{log(b)}{log(a)} = \frac{log_c(b)}{log_c(a)}}$ **provided that** $\boldsymbol{c \neq 1}$**.**

   For ex: $log_2(3) = \frac{log_{10}(3)}{log_{10}(2)}$ (Here we changed the base to 10)

<u>Some solved examples:</u>

Example 35. Convert the following exponential forms into logarithmic forms:

(i) $9^3 = 729$ **(ii)** $7^5 = 16807$ **(iii)** $2^{10} = 1024$

(iv) $10^{-3} = 0.001$ **(v)** $4^{-2} = 0.0625$ **(vi)** $5^{-4} = 0.0016$

(vii) $10^0 = 1$ **(viii)** $8^0 = 1$

Sol. (i) Given that $9^3 = 729$

$\Rightarrow \quad log_9 729 = 3 \qquad (by\ definition)$

which is required logarithmic form.

OR

Given that $9^3 = 729$

Taking logarithm on both sides, we get

$log9^3 = log729$

$\Rightarrow \quad 3\, log9 = log729 \qquad (used\ power\ property)$

$\Rightarrow \quad 3 = \frac{log729}{log9}$

$\Rightarrow \quad 3 = log_9 729 \qquad \left(used\ log_a b = \frac{log(b)}{log(a)}\right)$

which is required logarithmic form.

(ii) Given that $7^5 = 16807$

$\Rightarrow \quad log_7 16807 = 5 \qquad (by\ definition)$

which is required logarithmic form.

OR

Given that $7^5 = 16807$

Taking logarithm on both sides, we get

$$log 7^5 = log 16807$$

$\Rightarrow \quad 5\ log 7 = log 16807 \qquad (used\ power\ property)$

$\Rightarrow \quad 5 = \frac{log 16807}{log 7}$

$\Rightarrow \quad 5 = log_7 16807 \qquad \left(used\ log_a b = \frac{log(b)}{log(a)}\right)$

which is required logarithmic form.

(iii) Given that $2^{10} = 1024$

$\Rightarrow \quad log_2 1024 = 10$

which is required logarithmic form.

(iv) Given that $10^{-3} = 0.001$

$\Rightarrow \quad log_{10} 0.001 = -3$

which is required logarithmic form.

(v) Given that $4^{-2} = 0.0625$

$\Rightarrow \quad log_4 0.0625 = -2$

which is required logarithmic form.

(vi) Given that $5^{-4} = 0.0016$

$\Rightarrow \quad log_5 0.0016 = -4$

which is required logarithmic form.

(vii) Given that $10^0 = 1$

$\Rightarrow \quad log_{10} 1 = 0$

which is required logarithmic form.

(viii) Given that $8^0 = 1$

$\Rightarrow$ $log_8 1 = 0$

which is required logarithmic form.

Example 36. Convert the following logarithmic forms into exponential forms:

(i) $log_\pi 1 = 0$ **(ii)** $log_2 2048 = 11$ **(iii)** $log_4\left(\frac{1}{64}\right) = -3$

(iv) $log_3 243 = 5$ **(v)** $log_{10} 0.01 = -2$

Sol. (i) Given that $log_\pi 1 = 0$

$\Rightarrow$ $\pi^0 = 1$

which is required exponential form.

(ii) Given that $log_2 2048 = 11$

$\Rightarrow$ $2^{11} = 2048$

which is required exponential form.

(iii) Given that $log_4\left(\frac{1}{64}\right) = -3$

$\Rightarrow$ $4^{-3} = \frac{1}{64}$

which is required exponential form.

(iv) Given that $log_3 243 = 5$

$\Rightarrow$ $3^5 = 243$

which is required exponential form.

(v) Given that $log_{10} 0.01 = -2$

$\Rightarrow$ $10^{-2} = 0.01$

which is required exponential form.

Example 37. Evaluate the following:

(i) $log_2(8 \times 16)$ **(ii)** $log_4\left(\frac{16}{256}\right)$ **(iii)** $log_3 9^3$

(iv) $log_{10}\left(\frac{1}{10}\right)^8$ **(v)** $log_e\left(\frac{1}{e^{-7}}\right)$

Sol. (i) Given expression is

$$log_2(8 \times 16) = log_2(8) + log_2(16) \qquad (used\ product\ property)$$

$= log_2 2^3 + log_2 2^4$

$= 3. log_2 2 + 4. log_2 2$ $\quad$ $(used\ power\ property)$

$= 3 + 4 = 7$ $\quad$ $(used\ log_a a = 1)$

which is required solution.

OR

Given expression is

$log_2(8 \times 16) = log_2(128)$

$= log_2 2^7 = 7$ $\quad$ $(used\ log_a a^n = n)$

which is required solution.

(ii) Given expression is

$log_4\left(\frac{16}{256}\right) = log_4(16) - log_4(256)$ $\quad$ $(used\ quotient\ property)$

$= log_4 4^2 - log_4 4^4$

$= 2. log_4 4 - 4. log_4 4$ $\quad$ $(used\ power\ property)$

$= 2 - 4 = -2$ $\quad$ $(used\ log_a a = 1)$

which is required solution.

OR

Given expression is

$log_4\left(\frac{16}{256}\right) = log_4\left(\frac{4^2}{4^4}\right)$

$= log_4 4^{-2} = -2$ $\quad$ $(used\ log_a a^n = n)$

which is required solution.

(iii) Given expression is

$log_3 9^3 = log_3(3^2)^3$

$= log_3 3^6$

$= 6. log_3 3$ $\quad$ $(used\ power\ property)$

$= 6$ $\quad$ $(used\ log_a a = 1)$

which is required solution.

(iv) Given expression is

$$log_{10}\left(\frac{1}{10}\right)^8 = log_{10}\frac{1}{10^8}$$

$= log_{10}1 - log_{10}10^8$ $\quad$ $(used\ quotient\ property)$

$= 0 - 8.\,log_{10}10$ $\quad$ $(used\ log_a 1 = 0)$

$= -8$ $\quad$ $(used\ log_a a = 1)$

which is required solution.

(v) Given expression is

$$log_e\left(\frac{1}{e^{-7}}\right) = log_e e^7$$

$= 7.\,log_e e$

$= 7$ $\quad$ $(used\ log_a a = 1)$

which is required solution.

Example 38. Change the base of $log_2 3$ to 10 i.e. common logarithm**.**

Sol. Given expression is $log_2 3$

$$= \frac{log_{10}(3)}{log_{10}(2)}$$

Example 39. Change the base of $log_7 5$ to 5**.**

Sol. Given expression is $log_7 5$

$$= \frac{log_5(5)}{log_5(7)} = \frac{1}{log_5(7)}$$

Example 40. Solve the equation $log_2(x+1) = log_2(x) + log_2(2x+1)$ for x.

Sol. Given equation is

$$log_2(x+1) = log_2(x) + log_2(2x+1)$$

$\Rightarrow$ $log_2(x+1) = log_2\big(x.(2x+1)\big)$

$\Rightarrow$ $log_2(x+1) = log_2(2x^2 + x)$

$\Rightarrow$ $x + 1 = 2x^2 + x$ $\quad$ $(used\ log_a b = log_a c\ \ iff\ \ b = c)$

$\Rightarrow$ $2x^2 = 1$

$\Rightarrow$ $x^2 = \frac{1}{2}$

$\Rightarrow$ $x = \pm\sqrt{\frac{1}{2}}$

But x can't be negative as it is the argument of logarithm. Therefore, $= \sqrt{\frac{1}{2}}$.

Example 41. Solve the equation $log_5 a^2 = 1$ for a.

Sol. Given equation is

$$log_5 a^2 = 1$$

$$\Rightarrow \quad log_5 a^2 = log_5 5$$

$$\Rightarrow \quad a^2 = 5$$

$$\Rightarrow \quad a = \pm\sqrt{5}$$

Example 42. Solve the equation $log_{\frac{1}{2}}(y^2 - 1) = -1$ for y.

Sol. Given equation is

$$log_{\frac{1}{2}}(y^2 - 1) = -1$$

$$\Rightarrow \quad log_{\frac{1}{2}}(y^2 - 1) = -log_{\frac{1}{2}}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad log_{\frac{1}{2}}(y^2 - 1) = log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{-1}$$

$$\Rightarrow \quad log_{\frac{1}{2}}(y^2 - 1) = log_{\frac{1}{2}} 2$$

$$\Rightarrow \quad (y^2 - 1) = 2$$

$$\Rightarrow \quad y^2 = 3$$

$$\Rightarrow \quad y = \pm\sqrt{3}$$

Example 43. Prove that $log_b a \,.\, log_c b \,.\, log_a c = 1$ where a, b and c are positive and are not equal to 1.

Sol. $log_b a \,.\, log_c b \,.\, log_a c$

$$= \frac{loga}{logb} \cdot \frac{logb}{logc} \cdot \frac{logc}{loga} \quad (used\ change\ of\ base\ property)$$

$$= 1$$

Hence proved.

Example 44. Prove that $2log_2 4 + log_2 9 - log_2 6 = log_2 24$.

Sol. $2log_2 4 + log_2 9 - log_2 6$

$$= log_2 4^2 + log_2 9 - log_2 6$$

$$= log_2 16 + log_2 9 - log_2 6$$

$= log_2(16 \times 9) - \ log_2 6$

$= log_2\left(\frac{16\times9}{6}\right)$

$= log_2 24$

Hence proved.

Example 45. Prove that $log_{10}12 - 2log_{10}4 + \ 2log_{10}6 = log_{10}27$.

Sol. $log_{10}12 - 2log_{10}4 + \ 2log_{10}6$

$= log_{10}12 - log_{10}4^2 + \ log_{10}6^2$

$= log_{10}12 - log_{10}16 + \ log_{10}36$

$= log_{10}\left(\frac{12}{16}\right) + \ log_{10}36$

$= log_{10}\left(\frac{12}{16} \times 36\right)$

$= log_{10}27$

Hence proved.

EXERCISE -IV

1. Give the examples of following:

   (i) Product property (ii) Quotient property (iii) Power property.

2. Give the examples for following:

   (i) $log_a(b.c) \neq log_a b \times log_a c$

   (ii) $log_a\left(\frac{b}{c}\right) \neq \frac{log_a(b)}{log_a(c)}$

3. Convert the following exponential forms into logarithmic forms:

   (i) $5^5 = 3125$ (ii) $(0.2)^3 = 0.008$ (iii) $3^4 = 81$

   (iv) $2^{-3} = 0.125$ (v) $16^{-1} = 0.0625$ (vi) $e^0 = 1$

4. Convert the following logarithmic forms into exponential forms:

   (i) $log_{100}1 = 0$ (ii) $log_{10}0.0001 = -4$ (iii) $log_e\left(\frac{1}{e}\right)^3 = -3$

   (iv) $log_2 128 = 7$ (v) $log_{10}1000 = 3$

5. Evaluate the following:

   (i) $log_{10}\left(\frac{1}{10^{-12}}\right)$ (ii) $log_7\left(\frac{7^5}{49}\right)$ (iii) $log_3(3^5 \times 9^3)$

(iv) $log_e\left(\frac{1}{e}\right)^{50}$ (v) $log_2\left(\frac{1}{1024}\right)$

6. Change the base of $log_5 9$ to $'e'$ i.e. natural logarithm.

7. Change the base of $log_7 19$ to $'10'$ i.e. common logarithm.

8. Change the base of $log_3 11$ to 11.

9. Solve the following equation for x:

   (i) $log(2x^2 - 4) = log(2x) + log(x - 1)$

   (ii) $log(x^2) = log(40) - 3log2$

   (iii) $log(x) + log(6 + x) = log(16)$

   (iv) $log(x^2) = log(5x - 4)$

10. Prove that the following

   (i) $3log_2 4 + 4log_2 3 = log_2 5184$

   (ii) $3log4 - 2log2 + 3log5 = log80 + log25$

   (iii) $log_e e^6 + log_e\left(\frac{1}{e}\right)^3 = 3$

   (iii) $5log_2 2 - 2log_2 3 + log_2 18 = 6$

ANSWERS

3. (i) $log_5 3125 = 5$ (ii) $log_{0.2} 0.008 = 3$ (iii) $log_3 81 = 4$

   (iv) $log_2 0.125 = -3$ (v) $log_{16} 0.0625 = -1$ (vi) $log_e 1 = 0$

4. (i) $100^0 = 1$ (ii) $10^{-4} = 0.0001$ (iii) $e^{-3} = \left(\frac{1}{e}\right)^3$

   (iv) $2^7 = 128$ (v) $10^3 = 1000$

5. (i) 12 (ii) 3 (iii) 11 (iv) -50 (v) -10

6. $\frac{log_e 9}{log_e 5}$

7. $\frac{log_{10} 19}{log_{10} 7}$

8. $\frac{1}{log_3 11}$

9. (i) 2 (ii) $\pm\sqrt{5}$ (iii) 2 (iv) 1 , 4

1.5 DETERMINANTS AND MATRICES

Determinant : The arrangement of n^2 elements between two vertical lines in n rows and n–columns is called a determinant of order n and written as

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{vmatrix}_{n\times n}$$

Here a_{11}, a_{12}, a_{13}, … a_{nn} are called elements of determinant.

The horizontal lines are called rows and vertical lines are called columns.

Here $a_{11} \quad a_{12} \quad \dots \quad a_{1n} \rightarrow R_1$ (First Row)

$a_{21} \quad a_{22} \quad \dots \quad a_{2n} \rightarrow R_2$(IInd Row)

and

$$\begin{matrix} a_{11} & & a_{12} \\ a_{21} & & a_{22} \\ \dots & & \dots \\ a_{n1} & & a_{n2} \\ \downarrow & & \downarrow \\ \text{Ist column} & & \text{2nd column} \\ (C_1) & & (C_2) \end{matrix}$$

Determinant of order 2: The arrangement of 4 elements in two rows and two columns between two vertical bar is called a determinant of order 2.

i.e.

$$\begin{matrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{matrix} \rightarrow R_1\,(\text{first row}) \\ \rightarrow R_2\,(\text{2nd row}) \end{matrix} \\ \begin{matrix} C_1 & C_2 \\ (\text{1st}) & (\text{2nd}) \\ \text{Col.} & \text{Col.} \end{matrix} & \end{matrix}$$

Here a_{11}, a_{12}, a_{21}, a_{22} are called elements of determinant

Value of Determinant of order 2 :

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$ Multiplying diagonally with –sign in downward arrow

$= a_{11}\, a_{22} - a_{21}\, a_{12}$

Example 46. Solve $D = \begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix}$.

Sol. $D = \begin{vmatrix} 2 & 5 \\ 3 & 9 \end{vmatrix} = 2(9) - 3(5) = 18 - 15 = 3$

Example 47. Find the value of, $D = \begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$.

Sol. $D = \begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} = \sin^2\theta - (-\cos^2\theta)$ = $\sin^2\theta + \cos^2\theta$ = 1.

Determinant of 3rd Order : The arrangement of 3 × 3 = 9 elements between two vertical bars in 3–rows and 3 columns is called a determinant of order 3.

i.e.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{3} \end{vmatrix} \begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix}$$
$$\quad C_1 \quad C_2 \quad C_3$$

Here a_{11}, a_{12}, a_{13}, …, a_{33} are called elements of the determinant

Note : $a_{ij} \to$ element of i^{th} row and j^{th} column

$a_{mn} \to$ element of m^{th} row and n^{th} column

i.e. $a_{23} \to$ element of 2^{nd} row and 3^{rd} column.

Minor of an element

Definition : A minor of an element in a determinant in obtained by deleting row and column in which that element occurs.

Note: Minor of an element a_{ij} is denoted by M_{ij}

e.g. $$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of an element a_{11} = M_{11} = $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{eliminate}$$

eliminate

Minor of a_{12} = M_{12} = $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

Minor of a_{13} = $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{32} & a_{33} \end{vmatrix}$

Minor of a_{21} = M_{21} = $\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Similarly we can find minors of other elements.

Let Determinant of order 2 :

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Minor of a_{11} = M_{11} = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{22}$

Minor of a_{12} = M_{12} = a_{21}

Minor of a_{21} = M_{21} = a_{12}

Minor of a_{22} = M_{22} = a_{11}

Example 48. Find minor of each element in determinant $D = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$.

Minor of 1 = 4

Minor of 2 = 3

Minor of 3 = 2

Minor of 4 = 1

Example 49. Find minor of all the elements in the first row of the following determinant

$$D = \begin{vmatrix} 3 & 4 & -7 \\ -2 & 7 & 3 \\ 6 & -8 & 5 \end{vmatrix}$$

Sol. The elements in 1st row are 3, 4, –7.

(i) Minor of 3 = M_{11} (Deleting row and column in which element occurs)

i.e. $M_{11} = \begin{vmatrix} 7 & 3 \\ -8 & 5 \end{vmatrix} = 35 - (-24) = 59$

(ii) Minor of element 4 = $M_{12} = \begin{vmatrix} -2 & 3 \\ 6 & 5 \end{vmatrix} = -10 - 18 = -28$

(iii) Minor of 7 = $M_{13} = \begin{vmatrix} -2 & 7 \\ 6 & -8 \end{vmatrix}$ = 16 – 42 = –26.

Co–factor of an element

Co–factor of an element is a minor of that element with sign prefixed by the rule $(-1)^{i+j}$ where i and j are number of row and column in which that elements presents

e.g. $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

Co–factor of $a_{11} = (-1)^{1+1} M_{11} = a_{22}$

Co–factor of $a_{12} = (-1)^{1+2} M_{12} = -a_{21}$

Co–factor of $a_{21} = (-1)^{2+1} M_{21} = -a_{12}$

Co–factor of a_{22} = $(-1)^{2+2}$ M_{22} = a_{11}

Example 50. Find the co–factor of element 5 and 2 from the Determinant $D = \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix}$.

Sol. Co–factor of 5 = Co–factor of a_{12} = $(-1)^{1+2}$ M_{12} = (–1)(2) = –2

Co–factor of 2 = co–factor of a_{21} = $(-1)^{2+1}M_{21}$ = (–1)5 = –5.

Example 51. Find co–factor of a_{11} in $\begin{vmatrix} 2 & 7 \\ 5 & 3 \end{vmatrix}$.

Sol. Co–factor of a_{11} = $(-1)^{1+1}$ M_{11} = 3.

Example 52. Find co–factor of all element a_{11}, a_{13}, a_{32}, a_{23} in the determinant $D = \begin{vmatrix} 3 & 4 & -7 \\ -2 & 7 & 3 \\ 6 & -8 & 5 \end{vmatrix}$.

Sol. Co–factor of a_{11} = $(-1)^{1+1}$ M_{11} = $\begin{vmatrix} 7 & 3 \\ -8 & 5 \end{vmatrix} = 35 + 24 = 59$

Co–factor of a_{13} = $(-1)^{1+3}$ M_{13} = $\begin{vmatrix} -2 & 7 \\ 6 & -8 \end{vmatrix}$ = 16 – 42 = –26.

Co–factor of a_{23} = $(-1)^{2+3}$ M_{23} = $-\begin{vmatrix} 3 & 4 \\ 6 & -8 \end{vmatrix} = -(-24 - 24) = 48$

Co–factor of a_{32} = $(-1)^{3+2}$ M_{32} = $-\begin{vmatrix} 3 & -7 \\ -2 & 3 \end{vmatrix} = -(9 - 14) = 5$

Similarly, we can find the cofactors of the remaining elements.

Evaluation of Determinant of 3 × 3 by Laplace expansion method :

A Determinant of order 3 × 3 is given by

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} \rightarrow R_1 \\ \\ \\ \end{matrix}$$

Expanding the determinant by First Row

$= a_{11}$[Minor of a_{11}] – a_{12}[Minor of a_{12}] + a_{13}[Minor of a_{13}]

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Example 53. Solve the following determinant by Laplace expansion method

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 1 & 8 & 2 \end{vmatrix} \begin{matrix} \to R_1 \\ \\ \\ \end{matrix}$$

Sol. Given determinant is

$$D = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 2 \\ 1 & 8 & 2 \end{vmatrix} \begin{matrix} \to R_1 \\ \\ \\ \end{matrix}$$

Expanding by R_1, we get

= 3(Minor of 3) – 2(Minor of 2) + 1(Minor of 1)

$$= 3\begin{vmatrix} 5 & 2 \\ 8 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 4 & 5 \\ 1 & 8 \end{vmatrix}$$

= 3(10 – 16) – 2 (8 – 2) + 1(8 – 5)

= 3(–6) – 2(6) + 1(3)

= –18 –12+ 3 = –27

Example 54. Find x if $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$.

Sol. Given that $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$

$\Rightarrow \quad 16 - x^2 = 0 \quad \Rightarrow \quad x^2 = 16, \; x = \pm 4$

Example 55. If $\begin{vmatrix} 3 & 2 \\ x & 6 \end{vmatrix} = 0$, find value of x.

Sol. Given that $\begin{vmatrix} 3 & 2 \\ x & 6 \end{vmatrix} = 0$

$\Rightarrow \quad 18 - 2x = 0 \quad \Rightarrow \quad -2x = -18, \quad \Rightarrow \quad x = 9$

Example 56. Evaluate by Laplace expansion Method

$$D = \begin{vmatrix} 5 & -1 & 2 \\ 1 & 3 & 1 \\ 7 & 1 & 0 \end{vmatrix} \rightarrow R_1$$

Sol. Expanding by R_1

D = 5(Minor of 5) – (–1) [Minor of –1] + 2[Minor of 2]

$$D = 5\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 7 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}$$

D = 5(0 – 1) + 1(0 – 7) + 2(1 – 21)

D = –5 – 7 – 40 = –52

Solution of Equations by Cramer's Rule (having 2 unknown)

By Cramer's Rule, we can solve simultaneous equations with unknown using Determinants.

Solve the following equation by Cramer's rule

$$a_1x + b_1y = C_1 \qquad (1)$$

$$a_2x + b_2y = C_2 \qquad (2)$$

Solution of eqn**.** (1) and (2) by Cramer's Rule is

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}$$

where $$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$ (Replacing first column by constants)

$$D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$ (Replacing by 2^{nd} column by constants)

Note : The given equation have unique solution if $D \neq 0$.

Solution of Equation (in 3 unknown) by Cramer's Rule

Let three equations in three variable x, y, z be

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The solution of given equation is

$$x = \frac{D_1}{D}, \ y = \frac{D_2}{D}, \ z = \frac{D_3}{D}$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ (Determinant formed by coefficient of x, y, z)

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$ (obtained by replacing the 1st column by constant terms)

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$ (obtained by replacing the 2nd column by constant terms)

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$ (obtained by replacing the 3rd column by constant terms)

Now $$x = \frac{D_1}{D}, \; y = \frac{D_2}{D}, \; z = \frac{D_3}{D}$$

Note : (1) The equations have unique solution, if $D \neq 0$.

(2) The equations have an infinite number of solutions if $D = D_1 = D_2 = D_3 = 0$.

(3) The equations have no solution if $D = 0$ and any one of D_1, D_2 and D_3 is not zero.

Consistant : When a system of equations have a solution, then equations are said to be consistant.

Inconsistent : If equations have no solution, then equations are said to be inconsistent.

Example 57. Solve the system of equations using Cramer's rules :

$$x + 2y = 1$$

$$3x + y = 4$$

Sol. $$D = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5$$

$$D_1 = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = 1 - 8 = -7$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = 4 - 3 = 1$$

$\therefore$ $$x = \frac{D_1}{D} = \frac{-7}{-5} = \frac{7}{5}$$

$$y = \frac{D_2}{D} = \frac{1}{-5} = -\frac{1}{5}$$

Example 58. Solve by Cramer's rule

$$10x + 10y - z = -2$$

$$x + 5y + 2z = 0$$

$$x - 5y - z = 4$$

Sol. The coefficient determinant is

$$D = \begin{vmatrix} 10 & 10 & -1 \\ 1 & 5 & 2 \\ 1 & -5 & -1 \end{vmatrix} \rightarrow R_1 \qquad \text{Expanding by } R_1$$

$= 10(-5 + 10) - 10(-1 - 2) - 1(-5 - 5)$

$= 50 + 30 + 10 = 90$

$$D_1 = \begin{vmatrix} -2 & 10 & -1 \\ 0 & 5 & 2 \\ 4 & -5 & -1 \end{vmatrix} \rightarrow R_1 \qquad \text{Expanding by } R_1$$

$= -2(-5 + 10) - 10(0 - 8) - 1(0 - 20)$

$= -10 + 80 + 20 = 90$

$$D_2 = \begin{vmatrix} 10 & -2 & -1 \\ 1 & 0 & 2 \\ 1 & 4 & -1 \end{vmatrix} \rightarrow R_1 \qquad \text{Expanding by } R_1$$

$= 10(0 - 8) + 2(-1 - 2) - 1(4 - 0)$

$= -80 - 6 - 4 = -90$

$$D_3 = \begin{vmatrix} 10 & 10 & -2 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{vmatrix} \rightarrow R_1 \qquad \text{Expanding by } R_1$$

$= 10(20 + 0) - 10(4 - 0) - 2(-5 + 5)$

$= 200 - 40 + 20 = 180$

$\therefore$ $$x = \frac{D_1}{D} = \frac{90}{90} = 1$$

$$y = \frac{D_2}{D} = \frac{-90}{+90} = -1$$

$$z = \frac{D_3}{D} = \frac{180}{90} = 2$$

Hence $x = 1$, $y = -1$, $z = 2$.

Example 59. Apply Cramer's Rule to solve the equations:

$$x - 2y + z = 1$$

$$2x + 3y + 2z = 2$$

$$-x + y + 3z = -1$$

Sol. The coefficient Determinant is

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 2 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= 1(9 - 2) - (-2)(6 + 2) + 1(2 + 3)$$

$$= 7 + 16 + 5 = 28$$

$$D_1 = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 2 \\ -1 & 1 & 3 \end{vmatrix}$$

$$= 1(9 - 2) + 2(6 + 2) + 1(2 + 3)$$

$$= 7 + 16 + 5 = 28$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 3 \end{vmatrix}$$

$$= 1(6 + 2) - 1(6 + 2) + 1(-2 + 2)$$

$$= 8 - 8 + 0 = 0$$

$$D_3 = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= 1(-3 - 2) + 2(-2 + 2) + 1(2 + 3)$$

$$= -5 + 0 + 5 = 0$$

$$\therefore \qquad x = \frac{D_1}{D} = \frac{28}{28} = 1$$

$$y = \frac{D_2}{D} = \frac{0}{28} = 0$$

$$z = \frac{D_3}{D} = \frac{0}{28} = 0$$

Hence solution is x = 1, y = 0, z = 0.

EXERCISE -V

1. Find minors of all elements in $\begin{vmatrix} 7 & -3 \\ 4 & 2 \end{vmatrix}$.

2. Find minors and co–factors of all elements of determinant $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} 2 & 4 \\ -5 & 1 \end{vmatrix}$.

4. Evaluate $\begin{vmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{vmatrix}$

5. Expand the determinant by Laplace Method $D = \begin{vmatrix} 5 & -1 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix}$

6. Evaluate $\begin{vmatrix} 5 & -1 & 2 \\ 1 & 3 & 1 \\ 7 & 1 & 0 \end{vmatrix}$.

7. Find x if $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$.

8. Find x if $\begin{vmatrix} 4 & 3 & 9 \\ 3 & 2 & 7 \\ 1 & 4 & x \end{vmatrix} = 0$

9. Solve by Cramer's rule x + 3y = 4

   4x – y = 3

10. Solve by Determinant (Cramer's Rule) :

$$x + y + 2z = 4$$

$$2x - y + 2z = 9$$

$$3x - y - z = 2$$

11. Solve by Cramer's Rule :

$$x + y - z = 0$$

$$2x + y + 3z = 9$$

$$x - y + z = 2$$

12. Find Minors and Co–factors of 2nd row in elements of the Determinant.

$$\begin{vmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \\ 5 & 4 & -3 \end{vmatrix}$$

ANSWERS

1. $M_{11} = 2$, $M_{12} = 4$, $M_{21} = -3$, $M_{22} = 7$

2. $M_{11} = 3$, $M_{12} = 0$, $M_{21} = -4$, $M_{22} = 2$; $C_{11} = 3$, $C_{12} = 0$, $C_{21} = 4$, $C_{22} = 2$

3. 18   4. $\cos^2\boldsymbol{\theta}$ 5. –35   6. –52   7. $x = \pm 4$ 8. $x = -1$

9. $x = 1, y = 1$ 10. $x = 1, y = -1, z = 2$ 11. $x = 1, y = 1, z = 2$ 12. –4, –8, 4

MATRICES

Matrix: The arrangement of m × n elements in m–row and n–columns enclosed by a pair of brackets [ ] is called a matrix of order m × n. The matrix is denoted by capital letter A, B, C etc.

A matrix of order m × n is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

i.e. Matrix have m rows and n columns. In short we can write it as

$$A = [a_{ij}]$$

where i = 1, 2, 3, …, m ← row

j = 1, 2, 3, …, n ← columns

Note – Difference between a determinant & a matrix is that a determinant is always in square form [i.e. m = n], but matrix may be in square or in rectangular form. Determinant has a definite value, but matrix is only arrangement of elements with no value.

Order of Matrix : Number of rows × number of column

e.g. $A = \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}_{2\times 2}$ is a matrix of order 2 × 2

$B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \\ -2 & 0 \end{bmatrix}_{3\times 2}$ is a matrix of order 3 × 2

Types of Matrices :

(1) Square Matrix : A matrix is said to be a square matrix if number of rows of matrix is equal to number of columns of Matrix i.e. m = n.

For ex (i) $\begin{bmatrix} 2 & 0 \\ 5 & 7 \end{bmatrix}_{2\times 2}$ is a square matrix of order 2 × 2.

(ii) $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 1 \\ 0 & 5 & 6 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3 × 3

(2) Rectangular Matrix : A matrix where, number of rows is not equal to number of columns i.e. m ≠ n

e.g. $\begin{bmatrix} 1 & 4 & 7 \\ 2 & 0 & 6 \end{bmatrix}_{2\times 3}$ is a rectangular matrix of order 2 × 3

(3) Row Matrix : A matrix having one row and any number of columns is called row matrix e.g. (i) $A = [-3 \quad 2]_{1 \times 2}$ isarow matrix of order 1×2

(ii) $B = [5 \quad 7 \quad -2]_{1 \times 3}$ is a row matrix of order 1×3

(4) Column Matrix : A matrix having only one column and any number of rows is called column matrix.

e.g. $A = \begin{bmatrix} 2 \\ 5 \end{bmatrix}_{2\times1}$ order of matrix is 2×1

$B = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}_{3\times1}$ order of matrix is 3×1

(5) Diagonal Matrix : A matrix is to be a diagonal matrix if all non–diagonal elements are zero.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$

(6) Null Matrix : A matrix whose all elements are zero is called a null matrix. It is denoted by 0.

e.g. $0_{2x2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $0_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(7) Unit matrix : A diagonal matrix each of whose diagonal element is equal to unity is called unit matrix

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 2 and 3 respectively.

(8) Scalar Matrix : A diagonal matrix is said to be scalar matrix of all diagonal elements are equal.

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(9) Upper Triangular Matrix : A square matrix is which all the elements below the principle diagonal are zero is called an upper triangular matrix.

e.g. $\begin{bmatrix} 2 & 3 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(10) Lower Triangular Matrix : A square matrix in which all the elements above the principal diagonal are zero is called lower triangular matrix.

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 2 \end{bmatrix}$

(11) Equal Matrix : Two matrices are said to equal if they have same order and their corresponding elements are identical

For example $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

If $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_4 = 8$

(12) Transpose of Matrix : A matrix obtained by interchanging its rows and columns is called Transpose of the given matrix. Transose of A is denoted by A^T or A'.

e.g. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 5 & 6 \end{bmatrix}$

then $A^T = \begin{bmatrix} 2 & 2 \\ 3 & 5 \\ 1 & 6 \end{bmatrix}$

Note $(A^T)^T = A$.

(13) Symmetric Matrix : A matrix is said to be symmetric if it is equal to its transpose i.e. $A^T = A$.

For ex $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

(14) Skew–symmetric matrix : A square matrix is said to be skew–symmetric is

$$A^T = -A$$

Note : The diagonal elements of skew–symmetric matrix are always be zero.

For ex $$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$ is skew symmetric matrix

(15) Singular Matrix: A square matrix is said to be singular if | A | = 0. *i.e.* determinant, where | A | is the determinant of matrix A.

(16) Non–singular matrix : A matrix is said to be non–singular if $|A| \neq 0$.

Formation of a Matrix/Construction of Matrix :

Example 60. Construct a 2 × 2 matrix whose elements are $a_{ij} = i + j$.

Sol. We have

$$a_{ij} = i + j$$

Required matrix of 2 × 2 is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Example 61. Construct a matrix of 2 × 2 whose element is given by.$a_{ij} = \frac{(i+2j)^2}{2}$

Sol. We have

$$\therefore \quad a_{ij} = \frac{(i+2j)^2}{2}$$

$$a_{12} = \frac{(1+4)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2+2)^2}{2} = \frac{16}{4} = 8$$

$$a_{22} = \frac{(2+2(2))^2}{2} = \frac{36}{2} = 18$$

$\therefore \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$

Example 62. If $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ find the values of a and b.

Sol. Given matrices are equal, therefore their corresponding elements are identical

$\therefore \qquad a + b = 6 \qquad \Rightarrow \qquad a = 6 - b$

and $\qquad ab = 8$

$\Rightarrow \qquad (6 - b)b = 8$

$6b - b^2 = 8 \qquad \Rightarrow \qquad -b^2 + 6b - 8 = 0$

$\Rightarrow \qquad b^2 - 6b + 8 = 0 \quad \Rightarrow \qquad (b - 2)(b - 4) = 0$

$\Rightarrow \qquad b = 2$ and $b = 4$

If $\quad b = 2 \quad \Rightarrow \quad a = 6 - b = 6 - 2 = 4$

If $\quad b = 4 \quad \Rightarrow \quad a = 6 - b = 6 - 4 = 2$

$\therefore \qquad a = 2,\ b = 4$ & $a = 4,\ b = 2$

Example. 63. Find the value of x, y, z and a if $\begin{bmatrix} 2x+3 & x+2y \\ z+3 & 2a-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Sol. given matrices are equal

$\therefore \qquad 2x + 3 = 1 \qquad$ (i)

$x + 2y = 2 \qquad$ (ii)

$z + 3 = -1 \qquad$ (iii)

$2a - 4 = 3 \qquad$ (iv)

From eqn. (i) $\qquad 2x = 1 - 3 = -2$

$2x = -2 \Rightarrow \qquad x = -1$

Substitute $x = -1$ in (ii)

$$-1 + 2y = 2$$

$$2y = 3 \quad \Rightarrow \quad y = \frac{3}{2}$$

From eqn. (iii)

$$z + 3 = -1 \quad \Rightarrow \quad z = -4$$

From eqn. (iv)

$$2a - 4 = 3$$

$$2a = 7 \quad \Rightarrow \quad a = \frac{7}{2}$$

$$\therefore \quad x = -1,\ y = \frac{3}{2},\ z = -4,\ a = \frac{7}{2}$$

Algebra of Matrices : (Addition, Subtraction, and Multiplication of Matrices)

Addition of Matrices : If A and B are two matrices having same order, then their addition A + B is obtained by adding there corresponding elements

For example, If

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 8 \\ 7 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 7 \\ 0 & 6 \\ 2 & 3 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 2+5 & 5+7 \\ 6+0 & 8+6 \\ 7+2 & 0+3 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 6 & 14 \\ 9 & 3 \end{bmatrix}$$

Properties of Matrix addition :

If A, B and C are three matrix of same order then

(i) A + B = B + A (commutative law)

(ii) A + (B + C) = (A + B) + C [Associative law)

(iii) A + 0 = 0 + A, where **0** is null matrix

(iv) A + (–A) = (–A) + A = **0**

Here (–A) is called additive inverse of matrix A.

Subtraction of Matrices : If A and B are two matrices of same order, then A – B is obtained by subtracting element of B from the corresponding elements of A.

For example, Let $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 \\ 2 & 0 \end{bmatrix}$

then $A - B = \begin{bmatrix} 5-3 & 2-5 \\ 3-2 & 2-0 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

Note : A – B ≠ B – A.

Scalar Multiplication : The matrix obtained by multiplying each element of a given matrix by a scalar K.

If $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$, then scalar multiplication of A by the scalar 2 is given by

$$2A = 2\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 8 \end{bmatrix}$$

Multiplication of Two Matrices :

If A and B are two matrices, then their product AB is possible only **if number of columns in A is equal to number of rows in B.**

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$

Then $C = [c_{ij}]_{m \times p}$ is called the product of A and B.

Example 64. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} 6 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}_{2\times3}$ then find the product AB.

Sol. Here no. of column in A = No. of row in B.

Hence AB exist

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 6 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix}$$

R_1C_1, R_1C_2, R_1C_3, similarly R_2C_1, R_2C_2 & R_3C_3

$$AB = \begin{bmatrix} 2(6)+3(3) & 2(0)+3(1) & 2(1)+3(2) \\ 4(6)+5(3) & 4(0)+5(1) & 4(1)+5(2) \end{bmatrix}$$

$$= \begin{bmatrix} 12+9 & 0+3 & 2+6 \\ 24+15 & 0+5 & 4+8 \end{bmatrix} = \begin{bmatrix} 21 & 3 & 8 \\ 39 & 5 & 12 \end{bmatrix}$$

Note : $AB \neq BA$ (In general)

Properties of Multiplication of Matrices

(i) $AB \neq BA$ (in general)

(ii) A(B + C) = AB + AC

(iii) AI = A, where I is the unit matrix.

Power of a Matrix : If A is a square matrix *i.e.* number of rows = number of its columns then,

$A^2 = A.A$

$A^3 = A^2.A = A.A^2$

Similarly we can find other power of square matrix.

e.g. if $A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$, then find A^2

$$A^2 = A.A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4 & 2+0 \\ -2+0 & -4+0 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -2 & -4 \end{bmatrix}$$

Example 65. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, then find 5A and 3B.

Sol. We have $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, then

$$5A = 5\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 20 & 35 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 12 & 18 \end{bmatrix}$$

Example 66. If $A = \begin{bmatrix} 7 & 3 \\ -5 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 \\ 5 & 8 \end{bmatrix}$, then find A – B.

Sol. $$A - B = \begin{bmatrix} 7 & 3 \\ -5 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ -10 & -1 \end{bmatrix}$$

Example 67. If $A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, then find 2A – 3B.

Sol. We have

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix},\ 2A = 2\begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ -2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix},\ 3B = 3\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 9 & 6 \end{bmatrix}$$

$$2A - 2B = \begin{bmatrix} 10-6 & 6-(-3) \\ -2-9 & 2-6 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ -11 & -4 \end{bmatrix}$$

Example 68. If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$. Find 3X + Y.

Sol. $$3X = 3\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix}$$

$$3X + Y = \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 6+5 \\ -9+1 & 12-3 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$$

Example 69. If $A=\begin{bmatrix}2 & 3\\ 4 & 7\end{bmatrix}$, $B=\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}$, then find 2A + 3B + 5I, where I is a unit matrix of order 2.

Sol.

$$2A=2\begin{bmatrix}2 & 3\\ 4 & 7\end{bmatrix}=\begin{bmatrix}4 & 6\\ 8 & 14\end{bmatrix}$$

$$3B=3\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}=\begin{bmatrix}3 & 9\\ -6 & 15\end{bmatrix}$$

$$5I=5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$$

Now $$2A + 3B + 5I = \begin{bmatrix}4 & 6\\ 8 & 14\end{bmatrix}+\begin{bmatrix}3 & 9\\ -6 & 15\end{bmatrix}+\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$$

$$= \begin{bmatrix}4+3+5 & 6+9+0\\ 8-6+0 & 14+15+5\end{bmatrix}=\begin{bmatrix}12 & 15\\ 2 & 34\end{bmatrix}$$

Example 70. If $A=\begin{bmatrix}4 & 2\\ 8 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & 6\\ -4 & -12\end{bmatrix}$, find AB.

Sol.

$$AB = \begin{bmatrix}4 & 2\\ 8 & 4\end{bmatrix}\begin{bmatrix}2 & 6\\ -4 & -12\end{bmatrix}=\begin{bmatrix}R_1C_1 & R_1C_2\\ R_2C_1 & R_2C_2\end{bmatrix}$$

$$= \begin{bmatrix}8-8 & 24-24\\ 16-16 & 48-48\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Example 71. if $A=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}$, show that AB = BA.

Sol.

$$AB=\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}\begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2(3)+5(-1) & 2(-5)+5(2)\\ 1(3)+3(-1) & 1(-5)+3(2)\end{bmatrix}$$

$$= \begin{bmatrix}6-5 & -10+10\\ 3-3 & -5+6\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$BA = \begin{bmatrix}3 & -5\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ AB = BA

Example 72. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ -2 & 4 \end{bmatrix}$, show that $(A + B)^2 = A^2 + 2AB + B^2$.

Sol.

$$A + B = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -4 & 4 \end{bmatrix}$$

$$\text{LHS } (A + B)^2 = (A + B)(A + B) = \begin{bmatrix} -2 & 4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ -4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 8 \\ -8 & 0 \end{bmatrix} \quad (1)$$

$$A^2 = A.A. = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1-4 & 2+0 \\ -2+0 & -4+0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 \\ -2 & -4 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -3-4 & 2+8 \\ 6+0 & -4+0 \end{bmatrix} = \begin{bmatrix} -7 & 10 \\ 6 & -4 \end{bmatrix}$$

$$2AB = 2\begin{bmatrix} -7 & 10 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} -14 & 20 \\ 12 & -8 \end{bmatrix}$$

$$B^2 = B.B. = \begin{bmatrix} -3 & 2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} +9-4 & -6+8 \\ 6-8 & -4+16 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ -2 & 12 \end{bmatrix}$$

$\therefore$ RHS $A^2 + 2AB + B^2$

$$= \begin{bmatrix} -3 & 2 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} -14 & 20 \\ 12 & -8 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ -2 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} -3-14+5 & 2+20+2 \\ -2+12-2 & -4-8+12 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 24 \\ 8 & 0 \end{bmatrix}$$

$\therefore$ LHS $\neq$ RHS

Example 73. If $A = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 \\ 4 & 5 \end{bmatrix}$. Verify $(AB)^T = B^T A^T$.

Sol. Given matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 \\ 4 & 5 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $B^T = \begin{bmatrix} -3 & 4 \\ 5 & 5 \end{bmatrix}$.

$$AB = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -9+16 & 15+20 \\ -3+20 & 5+25 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 35 \\ 17 & 30 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 7 & 17 \\ 35 & 30 \end{bmatrix} \qquad (1)$$

$$B^T A^T = \begin{bmatrix} -3 & 4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -9+16 & -3+20 \\ 15+20 & 5+25 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 17 \\ 35 & 30 \end{bmatrix} \qquad (2)$$

From eqn.. (1) and (2)

$$(AB)^T = B^T A^T$$

Example 74. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, then find A^2.

Sol. $A^2 = A.A. = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+4 & 2+6 \\ 2+6 & 4+9 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}.$$

Example 75. If $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$. Show that $A^2 - 5A + 5I = 0$, where I is unit matrix of order 2.

Sol. We have $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 9+1 & 3+2 \\ 3+2 & 1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 5 & 10 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

LHS $A^2 - 5A + 5I$

$$\begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ 5 & 10 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10-15+5 & 5-5+0 \\ 5-5+0 & 5-10+5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 = \text{RHS}$$

EXERCISE-VI

1. Find the order of following matrices also find their type

(a) $[1 \quad 5 \quad 7]$ (b) $\begin{bmatrix} 6 \\ 9 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. If $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the value of x, y, z and w.

3. If $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$, find x, y, z.

4. Construct a 2 × 2 matrix whose element $a_{ij} = \frac{(1+j)^2}{2}$.

5. If $A = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$, find A + 2B.

6. Find the value of x, y, z and a if $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$.

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ -1 & 3 & 4 \end{bmatrix}$. Find $A^2 - 4A + 8I$, where I is unit matrix of order 3 × 3.

8. If A = $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. Show that AB = BA = I.

9. For the matrix A = $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$. Find the number a and b such that $A^2 + aA + bI = 0$, where I is unit matrix of 2 × 2.

10. If A = $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$. Show that AB = BA.

11. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$, then evaluate AB + 2I.

12. If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, then verify that

    (i) A(B + C) = AB + AC

    (ii) (B + C)A = BA + CA

13. If $A = \begin{bmatrix} 1 & 3 \\ x & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -3 \\ -2 & 3 \end{bmatrix}$. Verify that (A + B)C = AC + BC.

14. If A = $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, then find AB.

15. If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$. Find 3X + Y.

16. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, find :

(i) 2A + 3B – 4I, when I is a unit matrix

(ii) 3A – 2B

17. If A = $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$, find 2A + 3B – 5I, where I is unit matrix.

18. - The value of $\begin{vmatrix} 1 & 2 & 3 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$ is

(a) x^2 (b) 2x (c) 6x (d) x

19. - The value of $\begin{vmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{vmatrix}$ is

(a) 1 (b) a + b + c (c) 0 (d) abc

20 - Cofactors of the first row elements is the matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ are

(a) 3, -6, 3 (b) -3, 6, -3 (c) 3, 6, 3 (d) -3, -6, -3

21 - Two matrices A_{mxn} & B_{pxq} can be multiplied only when

(a) m = p (b) n = p (c) m = q (d) n = q

22 - If A = $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$ then A^2 is

(a) $\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ (d) None of these

ANSWERS

1. (a) order 1 × 3, Row matrix

   (b) order 3 × 1, Column matrix

   (c) order 2 × 2, Scalar matrix

   (d) order 2 × 2, Square matrix

   (e) order 2 × 3, Rectangular matrix

   (f) order 2 × 2, Null matrix

2. x = 2, y = 4, z = 1, and w = 3   3. x = 2, y = 1, z = 2

4. $\begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$ 5. $\begin{bmatrix} -3 & 5 \\ 1 & 3 \end{bmatrix}$ 6. x = –3, y = –2, z = 4, a = 3

7. $\begin{bmatrix} 12 & 15 & 17 \\ 8 & 61 & 57 \\ 14 & 16 & 26 \end{bmatrix}$ 11. $\begin{bmatrix} 4 & 1 \\ 13 & 16 \end{bmatrix}$ 14. $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+a^2 \end{bmatrix}$

15. $\begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$ 16.(i) $\begin{bmatrix} 3 & 15 \\ 20 & 28 \end{bmatrix}$, (ii) $\begin{bmatrix} 4 & 3 \\ 4 & 9 \end{bmatrix}$ 17. $\begin{bmatrix} 2 & 15 \\ 2 & 24 \end{bmatrix}$

18. (a)   19. (c)   20. (b)   21. (b)   22. (a)

1.6 PERMUTAITON AND COMBINATION

Before knowing about concept of permutation and combination first we must be familiar with the term factorial.

Factorial : Factorial of a positive integer ‘n’ is defined as :

$$n! = n \times (n-1) \times (n-2) \ldots 3 \times 2 \times 1$$

where symbol of factorial is ! or $\lfloor\underline{\ \ }$. For example

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Note $\quad 0! = 1$

Example 76. Evaluate

(i) $6!$ (ii) $3! + 2!$ (iii) $\frac{5!3!}{2!}$ (iv) $5! + 4!$

Sol.

(i) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

(ii) $3! + 2! = 3 \times 2 \times 1 + 2 \times 1 = 6 + 2 = 8$

(iii) $\frac{5!3!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1}{2 \times 1} = 360$

(iv) $5! - 3! = 5 \times 4 \times 3 \times 2 \times 1 - 3 \times 2 \times 1 = 120 - 6 = 114.$

Example 77. Evaluate (i) $\frac{n!}{(n-2)!}$ (ii) $(n - r)!$ when $n = 7, r = 3$.

Sol. $\frac{n!}{(n-2)!} = \frac{n \times (n-1) \times (n-2)(n-3) \ldots 3 \times 2 \times 1}{(n-2) \times (n-3) \ldots 3 \times 2 \times 1}$

$$= n(n-1) = n^2 - n$$

(ii) $(n - r)! = (7 - 3)! = 4! = 4 \times 3 \times 2 \times 1 = 24$

Example 78. Evaluate : (i) $\frac{8! - 6!}{3!}$ (ii) $\frac{2!}{4!} + \frac{7!}{5!}$

Sol. (i) $\frac{8 \times 7 \times 6! - 6!}{3!} = \frac{6![8 \times 7 - 1]}{3!}$

$$\frac{6! \times 55}{3!} = \frac{6 \times 5 \times 4 \times 3! \times 55}{3!} = 6600$$

(ii) $$\frac{2!}{4!}+\frac{7!}{5!}=\frac{2!}{4\times3\times2!}+\frac{7\times6\times5!}{5!}$$

$$=\frac{1}{12}+\frac{42}{1}=\frac{1+504}{12}=\frac{505}{12}$$

EXERCISE - VII

1. Compute $\lfloor 3 + \lfloor 6$.
2. Evaluate $\lfloor n-r$ where n = 8, r = 4.
3. Evaluate $\frac{10!}{8!3!}$.
4. Evaluate $\frac{7!-5!}{3!}$.
5. Evaluate product 3!.4!.7! and prove that 3! + 4! ≠ 7!.
6. Solve the equation (n + 1)! = 12 n!

ANSWERS

1. 726 2. 24 3. 15 4. 820 5. 725,760 6. n = 3

Permutation : It is the number of arrangements of 'n' different things taken 'r' at a time and is calculated by the formula,

$$^{n}P_{r}=\frac{n!}{(n-r)!}$$

Example 79. Evaluate $^{7}P_{4}$.

Sol. We know $^{n}P_{r}=\frac{n!}{(n-r)!}$

Put n = 7 and r = 4, we get

$$^{7}P_{4}=\frac{7!}{(7-4)!}=\frac{7!}{3!}=\frac{7\times6\times5\times4\times3\times2\times1}{3\times2\times1}=840$$

Example 80. Evaluate (i) $^{6}P_{6}$ (ii) $^{4}P_{1}$

Sol. (i) $^{6}P_{6}=\frac{6!}{(6-6)!}=\frac{6!}{0!}=\frac{6\times5\times4\times3\times2\times1}{1}=720$.

(ii) $^{4}P_{1}=\frac{4!}{(4-1)!}=\frac{4!}{3!}=\frac{4\times3!}{3!}=4$.

Combination : It is the grouping of 'n' different things taken 'r' at a time and is calculated by the formula.

$$^{n}c_{r}=\frac{n!}{(n-r)!r!}$$

Example 81. Evaluate : (i) $^{9}c_{5}$ (ii) $^{n}c_{0}$ (iii) $^{n}c_{n}$

Sol. (i) As $^{n}c_{r}=\frac{n!}{(n-r)!r!}$

Put n = 9, r = 5, we get

$$^{9}c_{5}=\frac{9!}{(9-5)!5!}=\frac{9!}{4!5!}=\frac{9\times8\times7\times6\times5!}{4\times3\times2\times1\times5!}=126$$

(ii) $^{n}c_{0}=\frac{n!}{(n-0)!0!}=\frac{n!}{n!\,0!}=1$ as 0! = 1

(iii) $^{n}c_{n}=\frac{n!}{(n-n)!n!}=\frac{n!}{0!n!}=1$

EXERCISE -VIII

1. Define permutation and combination with examples.

2. if n = 10, r = 4 then find value $\frac{n!}{(n-r)!}$.

3. Evaluate (i) $^{10}P_{2}$ (ii) $^{5}P_{5}$ (iii) $^{8}c_{3}$ (iv) $^{6}c_{0}$

4. Find n if ${}^{n}p_2 = 20$.

5. Find the value of ${}^{10}c_3 + {}^{10}c_4$.

6. If 11. ${}^{n}P_4 = 20.\ {}^{n-2}P_4$ then the value of n is

(a) 38 (b) 20 (c) 16 (d) None of these

7 - The value of 1.3.5……(2n -1) . 2^n equals

(a) $\frac{(2n)!}{2^n}$ (b) $\frac{n!}{2^n}$ (c) $\frac{(2n)!}{n!}$ (d) None of these

8 - If ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$, ${}^{n}C_{r+1} = 126$ then r is equal to

(a) 3 (b) 2 (c) 1 (d) None of these

9 - The value of 1 . 1! + 2 . 2! + 3 . 3! + 4 . 4! is

(a) 118 (b) 119 (c) 120 (d) None of these

ANSWERS

2. 5040 3. (i) 90 (ii) 1 (iii) 56 (iv) 1 4. 5 5. 330 6. (c) 7. (c)

8. (a) 9. (b)

1.7 BINOMIAL THEOREM

Binomial Theorem for Positive Integer: If n is any positive integer, then

$(x+a)^n = {}^{n}c_0 x^{n-0} a^0 + {}^{n}c_1 x^{n-1} a^1 + {}^{n}c_2 x^{n-2} a^2 + {}^{n}c_n x^{n-n} a^n$ is called Binomial expansion, where ${}^{n}c_0, {}^{n}c_1, {}^{n}c_2, \ldots, {}^{n}c_n$ are called Binomial co-efficients.

Features of Binomial Theorem :

(i) The number of terms in Binomial expansion is one more than power of Binomial expression.

(ii) In Binomial expansion the sum of indices of x and a is equal to n.

(iii) The value of Binomial co-efficient, equidistant from both ends is always same.

Application in Real Life :

In real life Binomial theorem is widely used in modern world areas such a computing, i.e. Binomial Theorem has been very useful such as in distribution of IP addresses. Similarly in nation's economic prediction, architecture industry in design of infrastructure etc.

Example 82. How may terms are there in binomial expansion of $(a + b)^7$.

Sol. The number of terms in binomial expansion of $(a + b)^7$ is (n + 1) where n = 7. So total number of terms in expansion = 8.

Example 83. Which of the binomial co–efficients have same value in $(x + a)^7$ ${}^7c_0, {}^7c_1, {}^7c_2, {}^7c_3, {}^7c_4, {}^7c_5, {}^7c_6, {}^7c_7$.

Sol. ${}^7c_0 = {}^7c_7 = 1,\quad {}^7c_1 = {}^7c_6 = 7$

${}^7c_2 = {}^7c_5 = 21\quad {}^7c_3 = {}^7c_4 = 35$

Example 84. Expand $(x + y)^7$ binomially.

Sol. $(x+y)^7 = {}^7c_0x^7y^0 + {}^7c_1x^6y^1 + {}^7c_2x^5y^2 + {}^7c_3x^4y^3 + {}^7c_4x^3y^4$

$$+{}^7c_5x^2y^5 + {}^7c_6x^1y^6 + {}^7c_7x^0y^7 \quad (1)$$

as ${}^7c_0 = {}^7c_7 = 1,\quad {}^7c_1 = {}^7c_6 = 7$

${}^7c_2 = {}^7c_5 = 21\quad {}^7c_3 = {}^7c_4 = 35$

so equation (1) becomes

$$(x+y)^7 = x^7 + 7x^6y^1 + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

EXERCISE - IX

1. State Binomial Theorem for n as a positive integer.
2. Write the number of terms in expansion of $(x + y)^{10}$.
3. Which of Binomial co–efficients in binomial expansion of $(a + b)^8$ have same value. Also evaluate.
4. Expand $(x + 2y)^5$ using Binomial theorem.

5. Expand $\left(x+\frac{1}{x}\right)^6$ using Binomial Theorem.

6. Expand $(2x – 3y)^4$ using Binomial Theorem.

7. Expand $(a^2 + b^3)^4$ using Binomial Theorem.

ANSWERS

2. 11 3. $^8c_0 = {}^8c_8 = 1, {}^8c_1 = {}^8c_7 = 8,\ {}^8c_2 = {}^8c_6 = 28,\ {}^8c_3 = {}^8c_5 = 56, {}^8c_4 = 70$

4. $(x+2y)^5 = x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

5. $\left(x+\frac{1}{x}\right)^6 = x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

6. $(2x-3y)^4 = 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

7. $(a^2 + b^3)^4 = a^8 + 4a^6b^3 + 6a^4b^6 + 4a^2b^9 + b^{12}$

General Term of Binomial Expression $(x + a)^n$ for positive integer 'n'

$T_{r+1} = {}^nc_r x^{n-r} a^r$ where $0 \le r \le n$

Generally we use above formula when we have to find a particular term.

Example 85. Find the 5^{th} term in expansion of $(x – 2y)^7$.

Sol. Compare $(x – 2y)^7$ with $(x + a)^n$

$\Rightarrow$ $x = x$, $a = –2y$, $n = 7$

$T_5 = T_{r+1}$ $\Rightarrow$ $r + 1 = 5$, $r = 4$

Using $T_{r+1} = {}^nc_r x^{n-r} a^r$

Put all values

$$T_{4+1} = {}^7c_4 x^{7-4}(-2y)^4$$

$$= \frac{7!}{3!4!} x^3 16y^4$$

$$= 560\ x^3\ y^4$$

Note : p^{th} term from end in expansion of $(x + a)^n$ is $(n - p + 2)^{th}$ term from starting.

To find Middle term in $(x + a)^n$ when n is positive integer

(a) when n is even positive integer,

$$\text{Middle term} = T_{\frac{n}{2}+1}$$

(b) When n is odd positive integer

$$\text{Middle term} = T_{\frac{n+1}{2}} \text{ and } T_{\frac{n+1}{2}+1}$$

Example 86. In Binomial expression (i) $(x + y)^{10}$ (ii) $(x + y)^{11}$. How many middle terms are there.

Sol. (i) In $(x + y)^{10}$

This binomial expression has only one Middle Term

i.e. $$T_{\frac{10}{2}+1} = T_6$$

(ii) In $(x + y)^{11}$

This Binomial expression has two middle terms

i.e. $$T_{\frac{11+1}{2}} \quad \text{and} \quad T_{\frac{11+1}{2}+1}$$

or T_6 and T_7.

Example 87. The 3^{rd} term from end in binomial expansion of $(x - 2y)^7$ is _____ term from starting.

Sol. Using formula $(n - p + 2)^{th}$ term.

Here n = 7, p = 3.

So 3^{rd} term end in binomial expansion of $(x - 2y)^7$ is $(7 - 3 + 2)^{th}$ term from starting, i.e. 6^{th} from starting.

Example 88. Find the middle term in expansion of $\left(\frac{x}{y}+\frac{y}{x}\right)^{12}$.

Sol. Middle term = $T_{\frac{12}{2}+1} = T_7$

Compare $\left(\frac{x}{y}+\frac{y}{x}\right)^{12}$ with $(x + a)^n$

$X = \frac{x}{y}$, $A = \frac{y}{x}$, $N = 12$

Using $T_{R+1} = {}^nC_r x^{n-r} a^r$

$T_{R+1} = T_7 \Rightarrow R = 6$

$T_{6+1} = {}^{12}C_6 \left(\frac{x}{y}\right)^{12-6} \left(\frac{y}{x}\right)^6$

$= 924 \left(\frac{x}{y}\right)^6 \left(\frac{y}{x}\right)^6$

$T_{6+1} = 924$

Example 89. Find 5th term from end in expansion of $\left(\frac{x^2}{2}-\frac{2}{x^3}\right)^9$.

Sol. Compare $\left(\frac{x^2}{2}-\frac{2}{x^3}\right)^9$ with $(x + a)^n$

$x = \frac{x^2}{2}$, $a = -\frac{2}{x^3}$, $n = 9$

5th term from end is $(n - p + 2)^{th}$ term from staring.

i.e. $(9 - 5 + 2)^{th}$ term from starting.

6th term from staring.

Using $T_{R+1} = {}^nC_r x^{n-r} a^r$

$T_{R+1} = T_6 \Rightarrow r + 1 = 6, \ r = 5.$

So, $T_{5+1} = {}^9C_5\left(\frac{x^2}{2}\right)^{9-5}\left(-\frac{2}{x^3}\right)^5$

$$= -126\frac{x^8}{16}\times\frac{32}{x^{15}}$$

$$T_{5+1} = \frac{-252}{x^7}$$

EXERCISE - X

1. Find 4th term of $\left(x+\frac{1}{x}\right)^7$ Binomially.

2. Find the 4th term of $\left(\frac{4x}{7}-y^2\right)^5$ Binomially.

3. Find middle term of $(x-2y)^7$ using Binomial theorem.

4. Find 5th term in Binomial expansion $\left(x^2+\frac{1}{x}\right)^7$.

5. Find 3rd term from end in Binomial expansion $(2x-3)^6$.

6. Find Middle Term in Binomial expansion $\left(\frac{2}{x}-\frac{x}{2}\right)^5$.

7. In Binomial expansion of $\left(\frac{4x}{7}-y^2\right)^5$. Find 4th term.

8. The total number of terms in the expansion of $(x+y)^{100}$ is

   (a) 100 (b) 200 (c) 101 (d) None of these

9. The coefficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^2$ is

   (a) $\frac{504}{259}$ (b) $\frac{405}{256}$ (c) $\frac{450}{263}$ (d) None of these

10. The constant term in the expansion of $\left(x-\frac{1}{x}\right)^{10}$ is

(a) 152 (b) -152 (c) 252 (d) -252

11 - In the expansion of $(3x + 2)^4$, the coefficient of middle term is

(a) 95 (b) 64 (c) 236 (d) 216

ANSWERS

1. $35x$ 2. $-\frac{160}{49}x^2y^6$ 3. (i) $-280x^4y^3$ (ii) $560x^3y^4$ 4. $35x^2$

5. $4860x^2$ 6. (i) $\frac{20}{x}$ (ii) $-5x$ 7. $-\frac{160}{49}x^2y^6$ 8. (c) 9. (a)

10. (d) 11. (d)

* * * * *