UNIT - 2

TRIGONOMETRY

Learning Objectives

- To understand angle, angle measurementsand theirconversions.
- To define T-ratios of standard and allied angles, sum, difference and product formulae.
- To apply trigonometric formulae in solving engineering problems.

Introduction: The word trigonometry is derived from two Greek words : trigono meaning 'a triangle' and metron meaning 'to measure'. Thus literally trigonometry means 'measurement of triangles'. In early stages of development of trigonometry, its scope lied in the measurement of sides and angles of triangles and the relationship between them. Though still trigonometry is largely used in that sense but of late it is also used in many other areas such as the science of seismology, designing electric circuits and many more areas.

2.1CONCEPT OF ANGLE

Definition : According to Euclid 'an angle is the inclination of a line to another line'. An angle may be of any magnitude and it may be positive or negative.

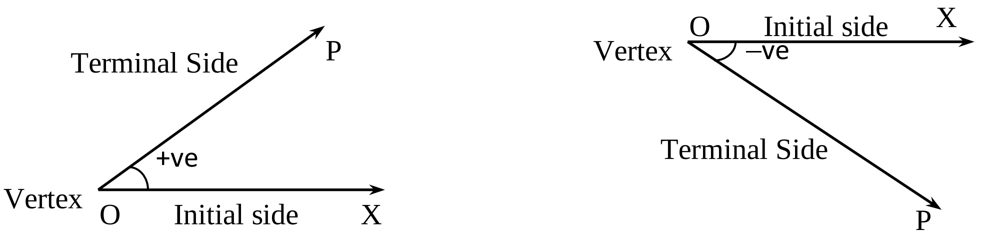


Fig. 2.1

Angle in any quadrant

Two mutually perpendicular straight lines XOX' and YOY' divide the plan of paper into four parts XOY, YOX', X'OY and Y'OX' which are called I, II, III, IV quadrants respectively.

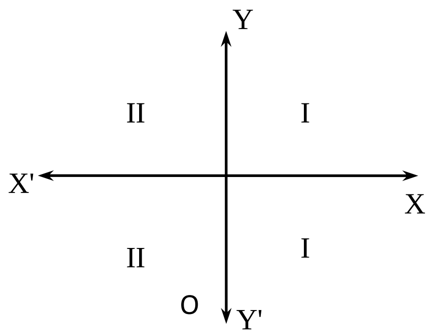


Fig. 2.2

Measurement of Angle

Sometimes different units are used to measure the same quantity. For example, time is measured in hours, minutes and seconds. In the same manner, we shall now describe three most commonly used units of measurement of an angle.

(i) Sexagesimal OR the English system

(ii) Centesimal OR the French system

(iii) Circular measure system.

Sexagesimal System (Degree measure)

In this system the unit of measurement is a degree. If the rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of revolution, then the angle is said to have a measure of one degree and is written as 1°.

A degree is further divided into minutes and a minute is divided into second $\left(\frac{1}{60}\right)^{th}$ of a degree is called a minute and $\left(\frac{1}{60}\right)^{th}$ of a minute is called a second. We can write as :

1 right angle = 90 degrees (written as 90°)

1 degree (1°) = 60 minutes (written as 60′)

1 minute (1′) = 60 seconds (written as 60")

Centesimal System : (Grade Measure) In this system a right angle is divided into 100 equal parts, each part called a grade. Each grade is further sub divided into 100 equal parts, called a minute and each minute is again divided into 100 equal parts, called a second. Thus, we have

1 right angle = 100 grades (written as 100^g)

1 grade (1^g) = 100 minutes (written as 100')

1 minute (1') = 100 second (written as 100")

Circular System (Radian measure) : In this system the unit of measurement is radian. A radian is the measure of an angle whose vertex is the centre of a circle and which cuts off an arc equal to the radius of the circle from the circumference:

One radian is shown below:

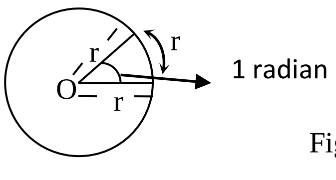


Fig 2.3

Thus a radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. One radius is denoted by as 1^c.

$$\pi^c = 180° \qquad \text{OR} \qquad \frac{\pi^c}{2} = 90°$$

We know that the circumferences of a circle of radius r is 2πr. Thus one complete revolution of the initial side subtends an angle of $\frac{2\pi r}{r}$ i.e., 2π radian.

Relation between three systems of an angle measurement

$$90° = 100^g = \frac{\pi^c}{2}$$

OR

$$180° = 200^g = \pi^c (\pi \text{ radius}) \qquad \left[\because \pi = \frac{22}{7}\right]$$

Example 1. Write the following angles in circular measure

(i) $75°$ (ii) $140°$

Sol.: (i) We know that

$$90^0 = \frac{\pi^c}{2}$$

$$1^0 = \frac{\pi^c}{2} \text{x} \frac{1}{90}$$

$$75^0 = \frac{\pi^c}{2} \text{x} \frac{1}{90} \text{ x } 75 = \frac{5\pi^c}{12}$$

(ii)

$$90^0 = \frac{\pi^c}{2}$$

$$1^0 = \frac{\pi^c}{2} \text{x} \frac{1}{90}$$

$$140^0 = \frac{\pi^c}{2} \text{x} \frac{1}{90} \text{ x } 140 = \frac{7\pi^c}{9}$$

Example 2. Find the centesimal measure of the angle whose radian measure are :

(i) $\frac{4\pi^c}{5}$ (ii) $\frac{3\pi^c}{10}$

Sol.(i) We know that

$$100^g = \frac{\pi^c}{2}$$

$$\frac{\pi^c}{2} = 100^g$$

$$\frac{\pi^c}{2} = 100$$

$\pi = 200$

$$\frac{4\pi}{5} = \frac{200}{5} \times 4 = 160^g$$

(ii) $$\frac{\pi^c}{2} = 100^g$$

$$\frac{\pi^c}{2} = 100$$

$\pi^c = 200$

$$\frac{3\pi^c}{10} = 200 \text{ x } \frac{3}{10} = 60^g$$

Example 3. Write the following angles in sexagesimal measure whose radians measures are:

(i) $\frac{\pi^c}{5}$ (ii) $\frac{\pi^c}{6}$

Sol : (i) We know that

$$90° = \frac{\pi^c}{2}$$

$$\pi^c = 180°$$

$$\frac{\pi^c}{5} = 180^0 \text{x } \frac{1}{5} = 36^0$$

(ii) We know that

$$\pi^c = 180°$$

$$\frac{\pi^c}{6} = 180^0 \text{x } \frac{1}{6} = 30^0$$

EXERCISE- I

1. Express in radians the followings angles

   (i) 45° (ii) 530° (iii) 40° 20'

2. Find the degree measures corresponding to the following radians measures

   (i) $\left(\frac{\pi}{8}\right)^c$ (ii) $\left(\frac{7\pi}{12}\right)^c$ (iii) $\left(\frac{3\pi}{4}\right)^c$

ANSWERS

1. (i) $\frac{\pi}{4}$ radians (ii) $\frac{53\pi}{18}$ radians (iii) $\frac{121\pi}{540}$ radians

2. (i) 22°30' (ii) 105° (iii) 42° 57' 17"

2.2 TRIGONOMETRIC RATIOS OF ANGLES

Trigonometric ratios are used to find the remaining sides and angles of triangles, when some of its sides and angles are given. This problem is solved by using some ratios of sides of a triangle with respect to its acute angles. These ratios of acute angles are called trigonometric ratios.

Sign of fundamental lines

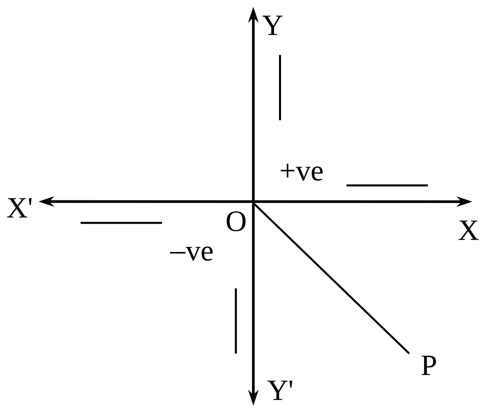
Let XOX' and YOY' be any two mutually perpendicular lines intersecting at O and dividing the plane into four parts

OX (towards the right) as +ve

OX' (towards the left) as –ve

OY (upwards) as +ve

OY'(downwards) as –ve



The revolving line OP is +ve in all its position.

Fig 2.4

Trigonometric Ratios: Let a revolving line OP starting from OX, trace an angle $\angle XOP = \theta$ where θ may be in any quadrant. From P draw perpendicular on XOX'

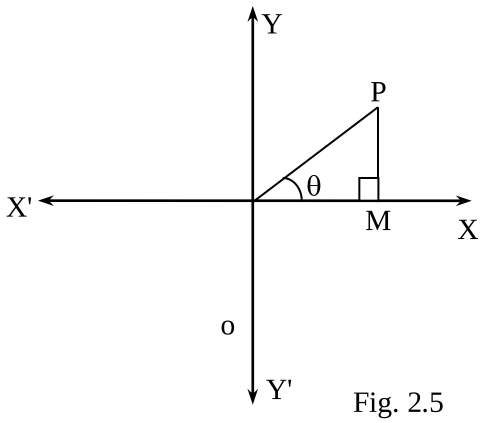


Fig. 2.5

So, in a right angled triangle, ratios are,

(i) $\frac{MP}{OP}\frac{(\text{Perpendicular})}{\text{Hypotenuse}}$ is called the sine of angle θ and written as sin θ.

(ii) $\frac{OM}{OP}\left(\frac{\text{Base}}{\text{Hypotenuse}}\right)$ is called the cosine of angle θ and written as cosθ.

(iii) $\frac{MP}{OM}\left(\frac{\text{Perpendicular}}{\text{Base}}\right)$ is called the tangent of angle θ and written as tan θ.

(iv) $\frac{OM}{MP}\left(\frac{\text{Base}}{\text{Perpendicular}}\right)$ is called cotangent of angle θ and written as cot θ.

(v) $\frac{OP}{OM}\left(\frac{\text{Hypotenuse}}{\text{Base}}\right)$ is called secant of angle θ and written as sec θ.

(vi) $\frac{OP}{MP}\left(\frac{\text{Hypotenus}}{\text{Perpendicular}}\right)$ is called cosecant of angle θ and written as cosec θ.

Relation between trigonometric ratios

(i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

(ii) $\sin^2\theta + \cos^2\theta = 1$

(iii) $\sec^2\theta - \tan^2\theta = 1$

(iv) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Signs of Trigonometric Ratios: The sign of various t-ratios in different quadrants are

(i) In first quadrant all the six t-ratios are positive.

(ii) In second quadrant only sin θ and cosec θ are positive and remaining t-ratios are negative.

(iii) In third quadrant only tan θ and cot θ are positive and remaining t-ratios are negative.

(iv) In fourth quadrant cosθ and sec θ are positive and remaining t-ratios are negative.

The revolving line OP is always positive.

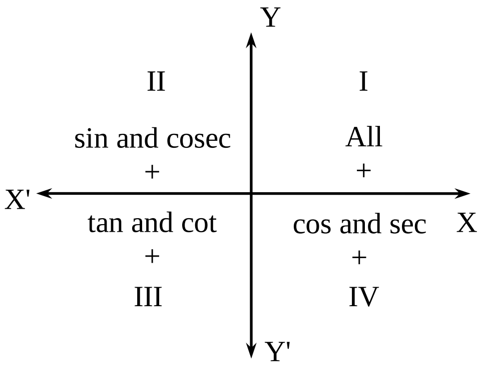


Fig. 2.6

Example4. In a ΔABC, right angle at A, if AB = 12, AC = 5 and BC = 13. Find the value of sin B, cos B and tan B.

Sol. In right angled ΔABC ;

Base = AB = 12,

Perpendicular = AC = 5

Hypotenus = BC = 13

$$\sin B = \frac{AC}{BC} = \frac{5}{13}$$

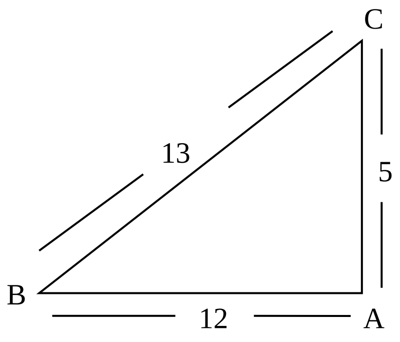


Fig. 2.7

$$\cos B = \frac{AB}{BC} = \frac{12}{13}$$

$$\tan B = \frac{AC}{AB} = \frac{5}{12}$$

Example 5. In a ΔABC, right angled at B if AB = 4, BC = 3, find the value of sin A and cos A.

Sol : We know by Pythagoras theorem

$AC^2 = AB^2 + BC^2$

$AC^2 = 4^2 + 3^2$

$= 16 + 9 = 25$

$AC^2 = (5)^2 \quad \therefore AC = 5$

$\therefore \quad \sin A = \frac{BC}{AC} = \frac{3}{5}$

$\cos A = \frac{AB}{AC} = \frac{4}{5}.$

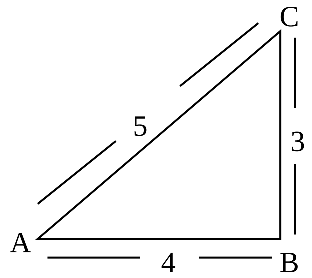


Fig. 2.8

Example 6. If sin A = $\frac{3}{5}$ find the value of cos A and tan A.

Sol : We know that sin A = $\frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{3}{5}.$

By Pythagoras theorem

$AC^2 = AB^2 + BC^2$

$5^2 = AB^2 + 3^2$

$25 = AB^2 + 9$

$AB^2 = 25 - 9 = 16$

$AB = 4$

$\cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{4}{5}$

$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{3}{4}$

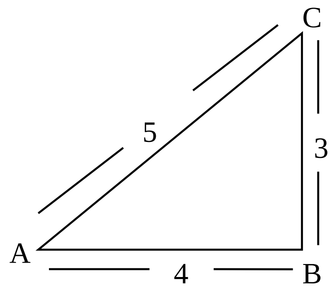


Fig. 2.9

Example7. If cosec A = $\sqrt{10}$. Find the values of sin A, cos A.

Sol : We have cosec A = $\frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{\sqrt{10}}{1}$

In a right angled triangle ABC.By Pythagoras theorem.

$AC^2 = AB^2 + BC^2$

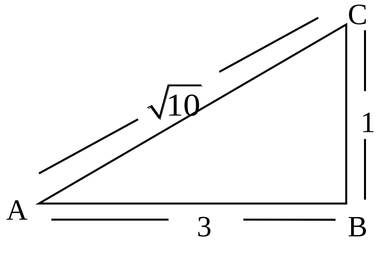


Fig. 2.10

$$(\sqrt{10})^2 = AB^2 + (1)^2$$

$10 = AB^2 + 1$

$AB^2 = 9 \qquad \therefore\ AB = 3$

$\therefore \qquad \sin A = \dfrac{\text{Perpendicular}}{\text{Hypotenuse}} = \dfrac{1}{\sqrt{10}}$

$\cos A = \dfrac{\text{Base}}{\text{Hypotenuse}} = \dfrac{3}{\sqrt{10}}$

EXERCISE-II

1. In a right triangle ABC, right angle at B, if sin A = $\frac{3}{5}$, find the value of cos A and tan A.

2. In a ΔABC, right angle at B, if AB = 12, BC = 5, find sin A and tan A.

3. In a ΔABC, right angled at B, AB = 24cm, BC 7cm. Find the value of sin A, cos A

4. If tan θ= $\frac{3}{5}$, find the value of $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

5. If 3 tan θ = 4, find the value of $\dfrac{4\cos\theta - \sin\theta}{2\cos\theta + \sin\theta}$.

6. If cot θ= $\frac{7}{8}$. Find the value of $\dfrac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$.

ANSWERS

1. $\cos A = \dfrac{4}{5},\ \tan A = \dfrac{3}{4}$ 2. $\sin A = \dfrac{5}{13},\ \tan A = \dfrac{5}{12}$ 3. $\dfrac{7}{25}, \dfrac{24}{25}$

4. $\dfrac{8}{2} = 4$ 5. $\dfrac{4}{5}$ 6. $\dfrac{49}{64}$.

T-Ratios of Standard Angles

The angles 0°, 30°, 60°, 90°, 180°, 270° and 360° are called standard angles. The value of 0°, 30°, 45°, 60° and 90° can be remembered easily with the help of following table:

Table 2.1

Angleθ	0°	30°	45°	60°	90°
$\sin\theta$	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}}=1$
$\cos\theta$	$\sqrt{\frac{4}{4}}=1$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{0}{4}}=0$
$\tan\theta$	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{2}}=1$	$\sqrt{\frac{3}{1}}=\sqrt{3}$	$\sqrt{\frac{4}{0}}=0$

T-ratios of Allied angles :

Allied angles : Two anglesare saidto be allied angles when their sum or differences is either zero or a multiple of 90°.

Complimentary angles : Two angles whose sum is 90° are called complement of each other. The angle θ and 90°–θ are complementary of each other.

Supplementary angles : Two angles whose sum is 180° are called supplementary of each other. The angles θ and 180°–θ are supplementary of each other.

The value of –θ, 90 ±θ, 180 ±θ, 270 ±θ and 360 ±θ can be remember easily by the following table.

Table 2.2

Angle	**Sin**	**Cos**	**Tan**	**Remarks**
–θ	sin(–θ) = –sin θ	cos(–θ) = cosθ	tan (–θ) = tanθ	
90 –θ	cosθ	sin θ	cot θ	Co–formulae apply
90 + θ	cosθ	–sin θ	–cot θ	Co–formulae apply
180 –θ	sin θ	–cosθ	– tan θ	
180 + θ	–sin θ	–cosθ	tan θ	
270 –θ	–cosθ	– sin θ	cot θ	Co–formulae apply

270 + θ	–cosθ	sin θ	–cot θ	Co–formulae apply
360 –θ	–sin θ	cosθ	–tan θ	
360 + θ	sin θ	cosθ	tan θ	

Example 8. Find the value of:

(i) sin 135° (ii) sin 300°.

Sol : (i) sin 135° = sin (90° + 45°) $\because$ sin(90° + **θ**) = cos**θ**

$$= \cos 45° = \frac{1}{\sqrt{2}}$$

OR

$$\sin 135° = \sin(180° - 45°)$$

$$= \sin 45° = \frac{1}{\sqrt{2}}$$

(ii) sin 300° = sin (360°– 60°) $\because$ sin(360°–**θ**) = –sin **θ**

$$= -\sin 60° = \frac{-\sqrt{3}}{2}$$

Example9. Evaluate

(i) tan 120° (ii) sin 150° (iii) cos 300° (iv) cot 225°

(v) sin(–690°)

Sol : (i) tan 120° = tan(90° + 30°) = –cot 30° = $-\sqrt{3}$

(ii) sin 150° = sin(180°– 30°) = sin 30° = $\frac{1}{2}$

(iii) cos 300° = cos(360°– 60°) = cos 60° = $\frac{1}{2}$

(iv) cot 225° = cot (180° + 45°) = cot 45° = 1

(v) sin (–690°) = –sin 690° = –sin (7 × 90° + 60°)

= cos 60° = + cos 60° = $\frac{1}{2}$

Example 10. Evaluate

(i) cos (–750°) (ii) sin (–240°) (iii) sin 765° (iv) cos 1050°

(v) tan (–1575°)

Sol : (i) cos (–750°) = +cos 750°

$= \cos(2 \times 360° + 30°)$

$= \cos 30° = \frac{\sqrt{3}}{2}$

(ii) $\sin(-240°) = -\sin 240°$

$= -\sin(180° + 60°)$

$= -\sin 60° = -\frac{\sqrt{3}}{2}$

(iii) $\sin 765° = \sin(2 \times 360° + 45°)$

$= \sin 45° = \frac{1}{\sqrt{2}}$

(iv) $\cos(1050°) = \cos(3 \times 360° - 30°)$

$= \cos(-30°) = \cos 30° = \frac{\sqrt{3}}{2}$

(v) $\tan(-1575°) = -\tan 1575°$

$= -\tan(4 \times 360° + 135°)$

$= -\tan 135°$

$= -\tan(180° - 45°)$

$= \tan 45° = 1$

Ex. 11. Evaluate the following :(i) $\frac{\cos 37°}{\sin 53°}$ (ii) $\sin 39° - \cos 51°$

Sol :(i) $\frac{\cos 37°}{\sin 53°} = \frac{\cos(90° - 53°)}{\sin 53°} = \frac{\sin 53°}{\sin 53°} = 1$

(ii) $\sin 39° - \cos 51°$

$= \sin(90° - 51°) - \cos 51°$

$= \cos 51° - \cos 51° = 0.$

EXERCISE-III

1. Evaluate the following :

(i) $\frac{\sin 41°}{\cos 49°}$ (ii) $\frac{\tan 54°}{\cot 36°}$ (iii) $\frac{\operatorname{cosec} 32°}{\sec 58°}$

2. Evaluate the following :

(i) cosec25°– sec 65° (ii) cot 34°– tan 56° (iii) $\frac{\sin 36°}{\cos 54°} - \frac{\sin 54°}{\cos 36°}$

3. Find the value of :

(i) sin 300° (ii) $\sin\left(-\frac{5\pi}{3}\right)$

ANSWERS

1. (i) 1 (ii) 1 (iii) 1 2. (i) 0 (ii) 0 (iii) 0

3. (i) $-\frac{\sqrt{3}}{2}$ (ii) $\frac{\sqrt{3}}{2}$

Addition and Subtraction Formulae

Addition Formulae **:**

Let a revolving line starting from OX, trace out an angle $\angle XOY = A$ and let it revolve further to trace an angle $\angle YOZ = B$. So that $\angle XOZ = A + B$ (Addition of angles A and B).

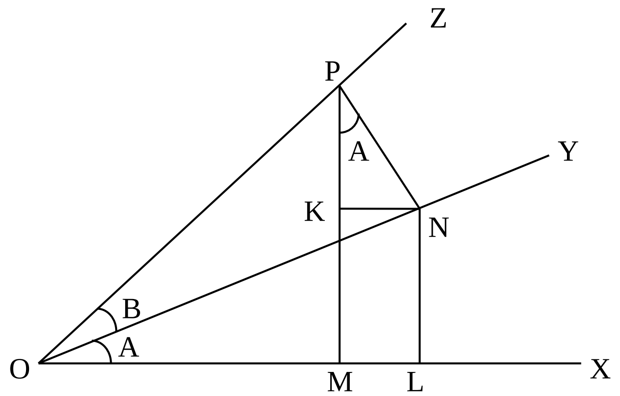


Fig. 2.11

Subtraction formulae **:**

Let a revolving line, starting from OX, trace out an angle $\angle XOY = A$ and let it revolve back to trace an angle $\angle YOZ = B$. So that $\angle XOZ = A - B$(Subtraction of Angle A and B)

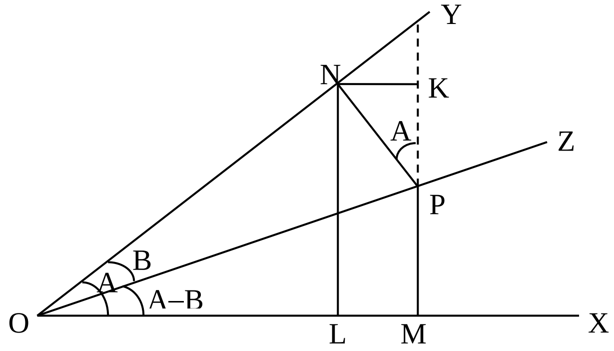


Fig. 2.12

A. Addition Formulae

(1) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(2) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(3) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(4) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

(5) $\tan(45 + A) = \dfrac{1 + \tan A}{1 - \tan A}$

B. Subtraction Formulae

(1) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

(2) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

(3) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(4) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

(5) $\tan(45 - A) = \dfrac{1 - \tan A}{1 + \tan A}$

Example 12. Evaluate

(i) sin 15°, cos 15°, tan 15° (ii) sin 75°, cos 75°, tan 75°

Sol : (i) (a) sin 15° = sin (45°– 30°)

= sin 45°cos 30°– cos 45° sin 30° [sin(A – B) = sin A cos B – cos A sin B]

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(b) cos 15° = cos (60°– 45°) [cos(A – B) = cosAcosB + sinAsinB]

= cos 60°cos 45° + sin 60° sin 45°

$$= \frac{1}{2}.\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{1}{4}(\sqrt{2}+\sqrt{6})$$

(c) tan 15° = tan (45°– 30°)

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ} \qquad \left[\tan (A - B) = \frac{\tan A - \tan B}{1+\tan A \tan B}\right]$$

$$= \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

(ii) (a) sin 75° = sin (45° + 30°)

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \qquad [\because \sin(A+B) = \sin A \cos B + \cos A \sin B]$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

(b) cos 75° = cos (45° + 30°)

= cos 45°cos 30°– sin 45° sin 30° [cos (A + B) = cos A cos B – sin A sin B]

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(c) tan 75° = tan (45° + 30°)

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1-\tan 45^\circ . \tan 30^\circ} \qquad \left[\tan (A + B) = \frac{\tan A + \tan B}{1-\tan A \tan B}\right]$$

$$= \frac{1+\frac{1}{\sqrt{3}}}{1-1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Example 13. Write down the values of :

(i) $\cos 68°\cos 8° + \sin 68° \sin 8°$ (ii) $\cos 50°\cos 10° - \sin 50° \sin 10°$

Sol :(i) $\cos 68°\cos 8° + \sin 68° \sin 8°$

$= \cos (68° - 8°)$ [$\cos(A - B) = \cos A \cos B + \sin A \sin B$]

$= \cos 60° = \frac{1}{2}$

(ii) $\cos 50°\cos 10° - \sin 50° \sin 10°$

$= \cos (50° + 10°)$ [$\cos(A + B) = \cos A \cos B - \sin A \sin B$]

$= \cos 60° = \frac{1}{2}$

Example 14. Prove that $\frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°} = \tan 56°$

Sol : L.H.S. = $\frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$

$$= \frac{1+\frac{\sin 11°}{\cos 11°}}{1-\frac{\sin 11°}{\cos 11°}}$$ [Dividing the num. and denom. by cos 11°]

$= \frac{1+\tan 11°}{1-\tan 11°} = \tan(45° + 11°)$ $\left[\tan (45^0 + A) = \frac{1+\tan A}{1-\tan A}\right]$

$= \tan 56°$ = R.H.S.

Example 15. Prove that tan3A tan2A tanA = tan3A – tan2A – tanA.

Sol : We can write, $\tan 3A = \tan (2A + A)$

$\Rightarrow$ $\tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$\Rightarrow$ $\tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$

$\Rightarrow$ $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$

Example 16. If tan A = $\sqrt{3}$, tan B = 2 – $\sqrt{3}$, find the value of tan (A – B).

Sol : Using the formula; $\tan(A-B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{\sqrt{3}+2-\sqrt{3}}{1-\sqrt{3}(2-\sqrt{3})}=\dfrac{2}{1-2\sqrt{3}+3}$

$$=\frac{2}{4-2\sqrt{3}}=\frac{2}{2(2-\sqrt{3})}=\frac{1}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$=\frac{2+\sqrt{3}}{4-3}=2+\sqrt{3}$$

$\therefore$ $\tan(A-B)=2+\sqrt{3}$

Example 17. If A and B are acute angles and sin A = $\dfrac{1}{\sqrt{10}}$, $\sin B=\dfrac{1}{\sqrt{5}}$. Prove that A+B = $\dfrac{\pi}{4}$.

Sol : Given, sin A = $\dfrac{1}{\sqrt{10}}$ and $\sin B=\dfrac{1}{\sqrt{5}}$

We know, cos A = $\sqrt{1-\sin^2 A}$ and $\cos B=\sqrt{1-\sin^2 B}$ [$\because$ A and B are acute angles]

$\Rightarrow$ cos A = $\sqrt{1-\dfrac{1}{10}}$ and cos B = $\sqrt{1-\dfrac{1}{5}}$

$\Rightarrow$ cos A = $\sqrt{\dfrac{9}{10}}$ and cos B = $\sqrt{\dfrac{4}{5}}$

$\Rightarrow$ cos A = $\dfrac{3}{\sqrt{10}}$ and cos B = $\dfrac{2}{\sqrt{5}}$

Now cos (A + B) = cos A cos B – sin A sin B

$$=\frac{3}{\sqrt{10}}.\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{10}}.\frac{1}{\sqrt{5}}=\frac{6}{\sqrt{50}}-\frac{1}{\sqrt{50}}$$

$$=\frac{5}{\sqrt{50}}=\frac{5}{5\sqrt{2}}=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$$

Hence A + B = $\dfrac{\pi}{4}$

Example 18. Prove that tan 70° = tan 20° + 2 tan 50°.

Sol : We can write; tan 70° = tan (20° + 50°)

$\Rightarrow$ tan 70° = $\frac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$ $\left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$

$\Rightarrow$ tan 70°– tan 20° tan 50° tan 70° = tan 20° + tan 50°

$\Rightarrow$ tan 70°– tan 20° tan 50° tan (90°– 20°) = tan 20° + tan 50°

$\Rightarrow$ tan 70°– tan 20° tan 50° cot 20° = tan 20° + tan 50°

$\Rightarrow$ tan 70°– tan 50° = tan 20° + tan 50° [∵ tan □. cot □ = 1]

$\Rightarrow$ tan 70° = tan 20° + 2 tan 50°

Example 19. Prove that tan 13 A – tan 9A – tan 4A = tan 13A tan 9A tan 4A.

Sol : We can write; tan 13A = tan (9A + 4A)

$$\tan 13A = \frac{\tan 9A + \tan 4A}{1 - \tan 9A \tan 4A} \qquad \left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$\Rightarrow$ tan 13A – tan 13A tan 9A tan 4A = tan 9A + tan 4A

$\Rightarrow$ tan 13A – tan 9A – tan 4A = tan 13A tan 9A tan 4A

Example 20. Prove that $\tan 2\theta - \tan\theta = \tan\theta \sec 2\theta$

Sol : L.H.S. = $\tan 2\theta - \tan\theta$

$$= \frac{\sin 2\theta}{\cos 2\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\cos 2\theta\cos\theta}$$

$$= \frac{\sin(2\theta - \theta)}{\cos 2\theta\cos\theta} = \frac{\sin\theta}{\cos 2\theta\cos\theta}$$ [sin(A – B) = sin A cos B –cos A cos B]

$$= \frac{\sin\theta}{\cos\theta\cos 2\theta} = \tan\theta\sec 2\theta.$$

Example 21. Proveby using trigonometric formulae that; tan 65° = tan 25° + 2 tan 40°.

Sol : We can write; 65° = 40° + 25°

tan 65° = tan (40° + 25°) = $\frac{\tan 40° + \tan 25°}{1 - \tan 40° \tan 25°}$

tan 65°– tan 40° tan 25° tan 65° = tan 40° + tan 25°

tan 65°– tan 40° tan 25° tan (90°– 25°) = tan 40° + tan 25°

$\tan 65° - \tan 40° \tan 25° \cot 25° = \tan 40° + \tan 25°$

$\tan 65° - \tan 40° = \tan 40° + \tan 25°$ $\quad [\because \tan\boldsymbol{\theta} \cot\boldsymbol{\theta} = 1]$

$\tan 65° = \tan 25° + 2 \tan 40°$

Hence proved

EXERCISE– IV

1. Evaluate (i) $\sin 105°$ (ii) $\cos 105°$ (iii) $\tan 105°$.

2. Evaluate : (i) $\sin 22° \cos 38° + \cos 22° \sin 38°$ (ii) $\dfrac{\tan 66° + \tan 69°}{1 - \tan 66° \tan 69°}$

3. Prove that :

   (i) $\sin 105° + \cos 105° = \cos 45°$

   (ii) $\cos A = -\dfrac{24}{25}$ and $\cos B = \dfrac{3}{5}$, where $\pi < A < \dfrac{3\pi}{2}; \dfrac{3\pi}{2} < B < 2\pi$; find sin (A + B) and cos (A + B)

   (iii) If $\tan A = \dfrac{5}{6}$ and $\tan B = \dfrac{1}{11}$. Show that $A + B = \dfrac{\pi}{4}$.

4. Prove that :

   (i) $\tan 28° = \dfrac{\cos 17° - \sin 17°}{\cos 17° + \sin 17°}$

   (ii) $\tan 58° = \dfrac{\cos 13° + \sin 13°}{\cos 13° - \sin 13°}$

5. If $\cos A = \dfrac{1}{7}$ and $\cos B = \dfrac{13}{14}$. Prove that $A - B = 60°$. A and B are acute angles.

6. Prove that :

   (i) $\tan 55° = \tan 35° + 2 \tan 20°$

   (ii) $\tan 50° = \tan 40° + 2 \tan 10°$

   (iii) $2 \tan 70° = \tan 80° - \tan 10°$

7. If $\tan A = \dfrac{1}{2}$ and $\tan B = \dfrac{1}{3}$. Show that $A + B = 45°$. Given that A and B are positive acute angles.

8. If $\tan A = \dfrac{a}{a+1}, \tan B = \dfrac{1}{2a+1}$. Show that $A + B = \dfrac{\pi}{4}$.

9. Prove that $\sqrt{3}\cos 23° - \sin 23° = 2\cos 53°$.

10. If $A + B = \dfrac{\pi}{4}$. Prove that (1 + tan A) (1 + tan B) = 2.

ANSWERS

1. (i) $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $\dfrac{1-\sqrt{3}}{2\sqrt{2}}$ (iii) $\dfrac{1+\sqrt{3}}{1-\sqrt{3}}$

2. (i) $\dfrac{\sqrt{3}}{2}$ (ii) 1

3. (i) $\dfrac{220}{221}, \dfrac{220}{221}$

Product formulae (Transformation of a Product into a Sum or Difference)

(i) 2 sin A cos B = sin (A + B) + sin (A – B)

(ii) 2 cos A sin B = sin (A + B) – sin (A – B)

(iii) 2 cos A cos B = cos (A + B) + cos (A – B)

(iv) 2 sin A sin B = cos (A – B) –cos (A + B)

Aid to memory

2 sin A cos B = sin (sum) + sin (difference)

2 cos A sin B = sin (sum) – sin (difference)

2 cos A cos B = cos (sum) + cos (difference)

2 sin A sin B = cos (difference) –cos (sum)

Example 22. Express the following as a sum or difference

(i) 2 sin 5x cos 3x (ii) 2 sin 4x sin 3x (iii) 8 cos 8x cos 4x

Sol :(i) 2 sin 5x cos 3x = sin (5x + 3x) + sin (5x – 3x)

$= \sin 8x + \sin 2x$

(ii) $2 \sin 4x \sin 3x = \cos (4x - 3x) - \cos (4x + 3x) = \cos x - \cos 7x.$

(iii) $8 \cos 8x \cos 4x = 4[2 \cos 8x \cos 4x] = 4[\cos (8x + 4x) + \cos (8x - 4x)]$

$= 4[\cos 12x + \cos 4x]$

Example 23. Prove that $\cos 20°\cos 40°\cos 60°\cos 80° = \frac{1}{16}$.

Sol : L.H.S. $= \cos 20°\cos 40° \frac{1}{2} \cos 80°$

$$= \frac{1}{2} (\cos 20°\cos 40°) \cos 80°$$

$$= \frac{1}{4} (2 \cos 20°\cos 40°) \cos 80°$$

$$= \frac{1}{4} (\cos 60° + \cos 20°) \cos 80°$$

$$= \frac{1}{4}\left[\frac{1}{2} + \cos 20^0\right]\cos 80^O$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80° + \cos 20° \cos 80°\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}\cos 80° + \frac{1}{2}(2\cos 20° \cos 80°)\right]$$

$$= \frac{1}{8} [\cos 80° + \cos 100° + \cos 60°]$$

As $\cos 100° = \cos (180° - 80°) = -\cos 80°$

$$= \frac{1}{8} [\cos 80° - \cos 80° + \frac{1}{2}] = \frac{1}{16} = \text{R.H.S.}$$

Example 24. Prove that, $\cos 10°\cos 50°\cos 70° = \frac{\sqrt{3}}{8}$.

Sol : LHS $= \frac{1}{2} [\cos 10°(2\cos 50°\cos 70°)]$

$$= \frac{1}{2} [\cos 10°(\cos 120 + \cos 20°)]$$

$= \frac{1}{2}$ [cos 10°($-\frac{1}{2}$ + cos 20°)]

$= -\frac{1}{4}\cos 10° + \frac{1}{4}(2\cos 10° \cos 20°)$

$= -\frac{1}{4}\cos 10° + \frac{1}{4}(\cos 30° + \cos 10°)$

$= -\frac{1}{4}\cos 10° + \frac{1}{4}\cos 30° + \frac{1}{4}\cos 10°$

$= \frac{1}{4}\cos 30° = \frac{1}{4}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$ = R.H.S

Example 25. Prove that sin 20° sin 40° sin 60° sin 80°= $\frac{3}{16}$.

Sol : L.H.S $= \frac{\sqrt{3}}{2}\sin 20° \sin 40° \sin 80°$ [$\sin 60^{O} = \frac{\sqrt{3}}{2}$]

$= \frac{1}{2}.\frac{\sqrt{3}}{2}\sin 20°(2\sin 80° \sin 40°)$

$= \frac{\sqrt{3}}{4}\sin 20°(\cos 40° - \cos 120°)$ ∵ 2sinAsinB=cos(A–B)–cos(A+B)

$= \frac{\sqrt{3}}{8}(2\sin 20° \cos 40°) + \frac{\sqrt{3}}{8}\sin 20°$

$= \frac{\sqrt{3}}{8}(\sin 60° - \sin 20°) + \frac{\sqrt{3}}{8}\sin 20°$

$= \frac{\sqrt{3}}{8}\sin 60° - \frac{\sqrt{3}}{8}\sin 20° + \frac{\sqrt{3}}{8}\sin 20°$

$= \frac{\sqrt{3}}{8}.\frac{\sqrt{3}}{2} = \frac{3}{16} =$ R.H.S.

Transformation of a sum or difference into a product formulae

(i) sin C + sin D = $2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(ii) sin C – sin D = $2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(iii) cos C + cos D = $2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(iv) cos C –cos D = $2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

Example 26. Express the following as product :

(i) sin 14 x + sin 2x (ii) cos 10°–cos 50° (iii) sin 80°– sin 20°

Sol :(i) sin 14x + sin 2x = $2\sin\frac{14x+2x}{2}\cos\frac{14x-2x}{2} = 2\sin 8x\cos 6x$.

(ii) cos 10°–cos 50° = $2\sin\frac{10°+50°}{2}\sin\frac{50°-10°}{2} = 2\sin 30°\sin 20°$

(iii) sin 80°– sin 20° = $2\cos\frac{80°+20°}{2}\sin\frac{80°-20°}{2} = 2\cos 50°\sin 30°$

Example 27. Prove that

(i) $\frac{\sin A+\sin B}{\cos A+\cos B} = \tan\frac{A+B}{2}$, (ii) $\frac{\cos 8x-\cos 5x}{\sin 17x-\sin 3x} = \frac{-\sin 2x}{\cos 10x}$

Sol : (i) L.H.S. = $\frac{\sin A+\sin B}{\cos A+\cos B} = \frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}$

$$= \frac{\sin\frac{A+B}{2}}{\cos\frac{A+B}{2}} = \tan\left(\frac{A+B}{2}\right) = \text{R.H.S.}$$

(ii) L.H.S. = $\frac{\cos 9x-\cos 5x}{\sin 17x-\sin 3x} = \frac{2\sin\frac{9x+5x}{2}\sin\frac{5x-9x}{2}}{2\cos\frac{17x+3x}{2}\sin\frac{17x-3x}{2}}$

$$= \frac{-2\sin 7x\sin 2x}{2\cos 10x\sin 7x} = \frac{-\sin 2x}{\cos 10x} = \text{R.H.S.}$$

Example 28. Prove that $\frac{\cos 4x+\cos 3x+\cos 2x}{\sin 4x+\sin 3x+\sin 2x} = \cot 3x$.

Sol : $\frac{\cos 4x+\cos 2x+\cos 3x}{\sin 4x+\sin 2x+\sin 3x}$

$$= \frac{2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2}+\cos 3x}{2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}+\sin 3x}$$

$$= \frac{2\cos 3x\cos x+\cos 3x}{2\sin 3x\cos x+\sin 3x}$$

$$= \frac{\cos 3x(2\cos x+1)}{\sin 3x(2\cos x+1)} = \frac{\cos 3x}{\sin 3x} = \cot 3x\,.$$

Example 29. Prove that

(i) $\dfrac{\sin A+\sin B}{\cos A+\cos B} = \tan\dfrac{A+B}{2}$ (ii) $\dfrac{\sin 7A+\sin 3A}{\cos 7A+\cos 3A} = \tan 5A\,.$

Sol :(i) L.H.S = $\dfrac{\sin A+\sin B}{\cos A+\cos B}$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} = \tan\frac{A+B}{2} = \text{R.H.S.}$$

(ii) L.H.S = $\dfrac{\sin 7A+\sin 3A}{\cos 7A+\cos 3A}$

$$= \frac{2\sin\left(\frac{7A+3A}{2}\right)\cos\left(\frac{7A-3A}{2}\right)}{2\cos\left(\frac{7A+3A}{2}\right)\cos\left(\frac{7A-3A}{2}\right)} \qquad \text{[Using CD formula]}$$

$$= \frac{\sin 5A\cos 2A}{\cos 5A\cos 2A} = \tan 5A = \text{R.H.S.}$$

Example 30. Prove that

(i) $\sin 47° + \cos 77° = \cos 17°$ (ii) $\sin 51° + \cos 81° = \cos 21°$

(iii) $\cos 52° + \cos 68° + \cos 172° = \cos 20° + \cos 100° + \cos 140°$

Sol : L.H.S = $\sin 47° + \cos 77°$

$= \sin 47° + \cos (90° - 13°) = \sin 47° + \sin 13°$

$$= 2\sin\frac{47°+13°}{2}\cos\frac{47°-13°}{2} \quad \left[\because \sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$= 2 \sin 30° \cos 17° = \frac{2}{2} \cos 17° = \cos 17° = \text{RHS} \quad [\because \sin 30 = \frac{1}{2}]$

(ii) L.H.S. = $\sin 51° + \cos 81° = \sin (90° - 39°) + \cos 81°$

$= \cos 39° + \cos 81° \qquad [\because \sin (90° - \square) = \cos \square]$

$$= 2\cos\frac{81°+39°}{2}\cos\frac{81°-39°}{2}\left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$= 2 \cos 60° \cos 21°$

$= 2 \times \frac{1}{2} \cos 21° = \cos 21°$ R.H.S. $\qquad [\because \cos 60° = \frac{1}{2}]$

L.H.S = $\cos 52° + \cos 68° + \cos 172°$

$$= 2\cos\frac{52°+68°}{2}\cos\frac{68°-52°}{2}+\cos 172°$$

$= 2 \cos 60° \cos 8° + \cos 172°$

$= 2 \times \frac{1}{2} \cos 8° + \cos (180° - 8°)$

$= \cos 8° - \cos 8° = 0 \qquad [\because \cos(180° - \theta) = -\cos\theta]$

R.H.S = $\cos 20° + \cos 100° + \cos 140°$

$$= 2\cos\frac{100°+20°}{2}\cos\frac{100°-20°}{2}+\cos 140°$$

$= 2 \cos 60° \cos 40° + \cos 140°$

$= 2 \times \frac{1}{2} \cos 40° + \cos (180° - 40°)$

$= \cos 40° - \cos 40° = 0 \qquad [\because \cos(180° - \theta) = -\cos\theta]$

$\therefore$ L.H.S = R.H.S

Example 31. $\cos A + \cos(120° - A) + \cos(120° + A) = 0$

Sol: L.H.S $= \cos A + \cos (120° - A) + \cos (120° + A)$

$$= \cos A + 2\cos\left(\frac{120°+A+120°-A}{2}\right)\cos\left(\frac{120°+A-120°+A}{2}\right)$$

$= \cos A + 2\cos 120°\cos A = \cos A + 2\left(-\frac{1}{2}\right)\cos A$

$= \cos A - \cos A = 0 = \text{R.H.S}$

EXERCISE- V

1. Express as sum or difference:

   (i) $2\sin 4\theta\cos 2\theta$ (ii) $2\sin\theta\cos 3\theta$

2. Prove that $\sin 10°\sin 30°\sin 50°\sin 70° = \frac{1}{16}$.

3. Prove that $\cos 20°\cos 40°\cos 80° = \frac{1}{8}$.

4. Prove that $\sin 10°\sin 50°\sin 60°\sin 70° = \frac{\sqrt{3}}{16}$.

5. Express the following as a product :

   (i) $\sin 7\theta + \sin 3\theta$ (ii) $\cos 5\theta + \cos 3\theta$

   (iii) $\sin 5\theta - \sin\theta$ (iv) $\cos 2\theta - \cos 4\theta$

6. Prove that : (i) $\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A$ (ii) $\frac{\sin 7x + \sin 3x}{\cos 7x + \cos 3x} = \tan 5x$

7. Prove that $\cos 28° - \sin 58° = \sin 2°$.

8. Prove that :

   (i) $\cos 52° = \cos 68° + \cos 172° = 0$

   (ii) $\sin 50° - \sin 70° = \sin 10° = 0$

9. Prove that $\sqrt{3}\cos 13° + \sin 13° = 2\sin 13°$.

10. $\frac{\sin 11A\sin A + \sin 7A\sin 3A}{\cos 11A\sin A + \cos 7A\sin 3A} = \tan 8A$.

ANSWERS

1. (i) $\sin 6\theta + \sin 2\theta$ (ii) $\sin 4\theta - \sin 2\theta$

5. (i) $2 \sin 5\theta \cos 2\theta$ (ii) $2 \cos 3\theta \sin 2$ (iii) $2 \cos 4\theta \cos\theta$ (iv) $2 \sin 3\theta \sin \theta$

T–Ratios of Multiple and Submultiple Angles

(i) $\sin 2A = 2 \sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$.

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$.

Remember

(i) sin(any angle) = 2 sin (half angle) cos (half angle)

(ii) cos (any angle) = $\cos^2$ (half angle) – $\sin^2$ (half angle)

= 2 $\cos^2$ (half angle) – 1

= 1 – 2 $\sin^2$ (half angle)

= $\dfrac{1-\tan^2(\text{half angle})}{1+\tan^2(\text{half angle})}$

(iii) tan(any angle) = $\dfrac{2\tan(\text{half angle})}{1-\tan^2(\text{half angle})}$

Remember

(i) $\sin^2$(any angle) = $\dfrac{1-\cos(\text{double the angles})}{2}$

(ii) $\cos^2$ (any angle) = $\dfrac{1+\cos(\text{double the angles})}{2}$

(iii) $\tan^2$ (any angle) = $\dfrac{1-\cos(\text{double the angles})}{1+\cos(\text{double the angles})}$

T–ratios of 3A in terms of those of A

(i) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(ii) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

2.3 APPLICATIONS OF TRIGONOMETRIC TERMS IN ENGINEERING PROBLEMS

Height and Distance: Trigonometry helps to find the height of objects and the distance between points.

Angle of Elevation : The angle of elevation is for objects that are at a level higher than that of the observer.

$\angle XOP$ is called angle of elevation.

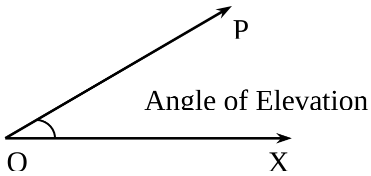


Fig. 2.13

Angle of Depression : The angle of depression is for objects that are at a level which is lower than that of the observer.

$\angle XOP'$ is called angle of depression.

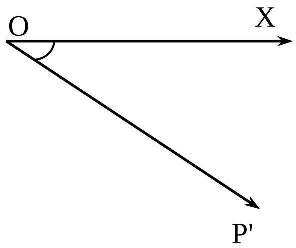


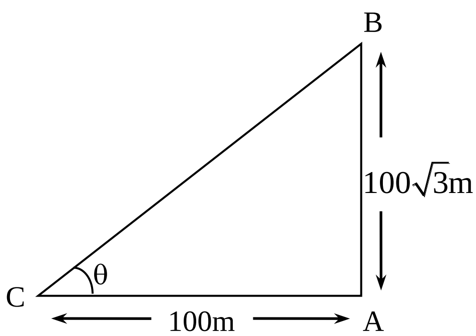
Fig. 2.14

Example 32. A tower is $100\sqrt{3}$ metres high. Find the angle of elevation of its top from a point 100 metres away from its foot.

Sol : Let AB the tower of height $100\sqrt{3}$ m and let C be a point at a distance of 100 metres from the foot of tower. Let θ be the angle of elevation of the top of the lower from point c.

In right angle ΔCAB

$$\frac{AB}{BC} = \tan\theta$$



$$\frac{100\sqrt{3}}{100} = \tan\theta \qquad \therefore$$

$\tan\theta = \sqrt{3} = \tan 60°$

Fig. 2.15

$\Rightarrow \qquad \theta = 60°,$

Hence the angle of elevation of the top of the tower from a point 100 metre away from its foot is 60°.

Example 33. A kite is flying at a height of 60 metres above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string assuming that there is no slack in the string.

Sol : Let A be the kite and CA be the string attached to the kite such that its one end is tied to a point C on the ground. The inclination of the string CA with the ground is 60°.

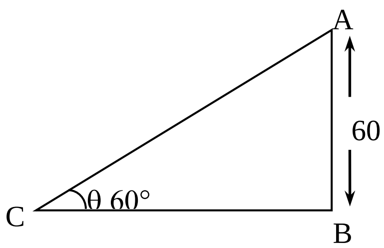


Fig. 2.16

In right angle ΔABC we have

$$\frac{AB}{AC} = \sin\theta$$

$$\frac{AB}{AC} = \sin 60°$$

$$\frac{60}{AC} = \frac{\sqrt{3}}{2} \qquad \text{or} \qquad AC = \frac{120}{\sqrt{3}} = 40\sqrt{3}\text{m}.$$

Hence the length of the string is $40\sqrt{3}$ metres.

Example 34. The string of a kite is 100 metres long and it make and angle of 60° with the horizontal. Find the height of the kite assuming that there is no slack in the string.

Sol : Let CA be the horizontal ground and let B be the position of the kite at a height h above the ground. The AB = h.

In right angle ΔCAB

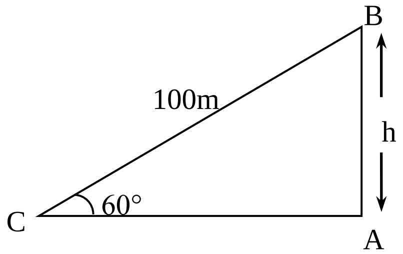


Fig. 2.17

$$\frac{AB}{CB} = \sin 60°$$

$$\frac{h}{100} = \frac{\sqrt{3}}{2} \qquad \text{or} \qquad h = 100\frac{\sqrt{3}}{2}$$

$\therefore \qquad h = 50\sqrt{3}$ metres

Hence the height of the kite is $50\sqrt{3}$ metres.

Example 35. A circus artist is climbing from the ground along a rope stretched from the top of a vertical pole and tied at a ground. The height of the pole is 12m and the angle made by the rope with the ground level is 30°. Calculate the distance covered by the artist in climbing to the top of the pole ?

Sol : Let vertical pole AB of height in metres and CB be the rope.

In right angle ΔCAB

$$\frac{AB}{CB} = \sin\theta = \sin 30°$$

$$\frac{12}{CB} = \frac{1}{2} \qquad \therefore \ CB = 24 \text{ m}$$

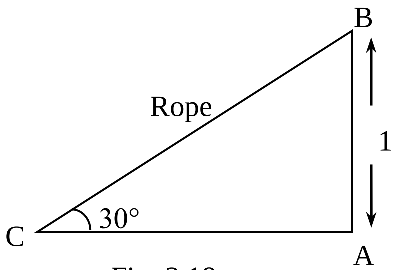


Fig. 2.18

Hence the distance covered by the circus artist is 24 m.

Example 36. Two polls of equal height stand on either side of a roadways which is 40 metres wide at a point in the roadway between the polls. The elevation of the tops of the polls are 60° and 30°. Find their height and the position of the point ?

Sol : let AB = 40 metres be the width at the roadway. Let AD = h, BC = h metres be the two polls. Let P be any point on AB at which the ngle of elevation of the tops are 60° and 30°.

Then $\angle APD = 60°$ and $\angle BPC = 30°$

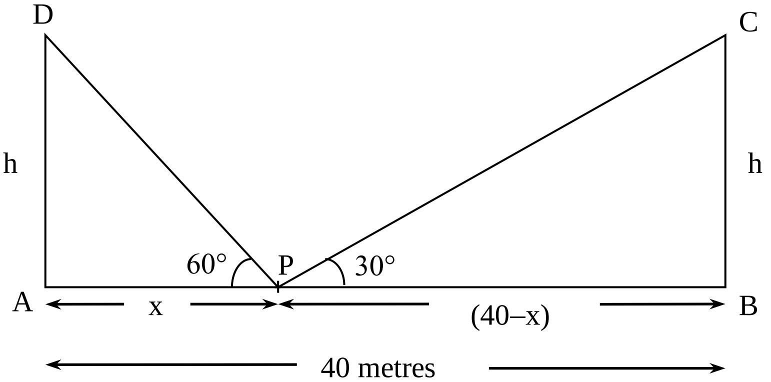


Fig. 2.19

Let AP = x $\quad\quad\therefore$ PB = 40 – x

Now from right angle Δ PBC

$$\frac{BC}{BP} = \tan 30° \quad \text{or} \quad \frac{h}{40-x} = \frac{1}{\sqrt{3}}$$

$$\sqrt{3}h = 40 - x \quad \ldots . . (i)$$

Again from right angle ΔPAD

$$\frac{AD}{PA} = \tan 60° \quad \text{or} \quad \frac{h}{x} = \sqrt{3}$$

$$\therefore \quad h = \sqrt{3}x \quad \ldots . . (ii)$$

Substituting the value of h in eqn**.** (i), we get

$$\sqrt{3} \times \sqrt{3}x = 40 - x \quad \text{or} \quad 3x = 40 - x$$

4x = 40 so $\quad$ x = 10 metres.

If x = 10 metres than h = $\sqrt{3}.10$ = 17.32 metres (height of polls)

Hence the point P divides AB in the ratio 1 : 3.

EXERCISE- VI

1. A tower stands vertically on the ground from a point on the ground, 20 m away from the foot of the tower, the angle of elevation of the top of the tower is 60°. What is the height of the tower ?
2. The angle of elevation of a ladder leaning against a wall is 60° and the foot of the ladder is 9.5m away from the wall. Find the length of the ladder.
3. A ladder is placed along a wall of a house such that upper end is touching the top of the wall. The foot of the ladder is 2m away from the wall and ladder is making an angle of 60° with the level of the ground. Find the height of the wall ?
4. A telephone pole is 10 m high. A steel wire tied to tope of the pole is affixed at a point on the ground to keep the pole up right. If the wire makes an angle 45° with the horizontal through the foot of the pole, find the length of the wire.

5. A kite is flying at a height of 75 metres from the ground level, attached to a string inclined at 60° to the horizontal. Find the length of the string to the nearest metre.

6. A vertical tower stands on a horizontal plane a is surmounted by a vertical flag–staff. At a point on the plane 70 metres away from the tower, an observer notices that the angle of elevation of the top and bottom of the flag–staff are respectively 60° and 45°. Find the height of the flag–staff and that of the tower.

7. A circus artist is climbing a 20 metre long rope which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole if the angle made by the rope with the ground level is 30°.

8. A person standing on the bank of a river, observer that the angle subtended by a tree on the opposite bank is 60°. When the retreats 20 metres from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river ?

9. The magnitude of a radian is

(a) 60^0 (b) $57^0\ 17'\ 44.8''$ nearly (c) $58^0\ 59'$ (d) None of these

10. Angular measurement of an angle is

(a) The number of degrees in an angle
(b) The number of radians in an angle
(c) The number of grades in an angle
(d) None of these

11. The angle subtended by an arc of 1 meter at the centre of a circle with 3 meter radius is

(a) 60^0 (b) 20^0 (c) $\frac{1}{3}$ (d) 3 radian

12. sin (A + B) . sin (A - B) =

(a) $\sin^2 A - \sin^2 B$ (b) $\cos^2 A - \cos^2 B$

(c) $\sin (A^2 - B^2)$ (d) $\sin^2 A - \cos^2 B$

13. The value of $\sin 60^0 \cos 30^0 + \cos 300^0 \sin 330^0$ is

(a) 1 (b) -1 (c) 0 (d) None of these

14. If $\cos \theta = -\frac{12}{13}$ then $\tan \theta$ is

(a) $-\frac{12}{13}$ but not $\frac{12}{15}$ (b) $-\frac{12}{5}$ and $\frac{12}{5}$

(c) $\frac{12}{5}$ and $-\frac{12}{5}$ (d) None of these

15. The value of $\tan 380^0 \cot 20^0$ is

(a) 0 (b) 1 (c) $\tan^2 20^0$ (d) $\cot^2 20^0$

16. The value of the expression $\frac{\sin 70^0}{\sin 110^0}$ is

(a) 2 (b) 0 (c) 1 (d) None of these

17. $\frac{\cos 17^0+\sin 17^0}{\cos 17^0-\sin 17^0}$ =

(a) tan 17^0 (b) tan 62^0 (c) tan 29^0 (d) None of these

18. A tower is $200\sqrt{3}$ m high then the angle of elevation of its top from a point 200m away from its foot is:

(a) 30^0 (b) 60^0 (c) 45^0 (d) None of these

ANSWERS

1. $20\sqrt{3}$ 2. 19 m 3. $2\sqrt{3}$m 4. 14.1 m5. 87 m

6. 51.24 m and 70 m 7. 10 metres 8. h = 17.32 m, breadth 9. (b) 10. (b)

11. (c) 12. (a) 13. (d) 14. (b) 15. (b) 16. (c)

17. (b) 18. (b)

UNIT 3

CO-ORDINATE GEOMETRY

Learning Objectives

- To understand and identify features of two dimensional figures; point, straight line and circle.
- To learn different forms of straight line and circle with different methods to solve them.
- Understand the basic concepts of two dimensional coordinate geometry with point, straight line and circle.

3.1 POINT

Cartesian Plane: Let XOX' and YOY' be two perpendicular lines. 'O' be their intersecting point called origin. XOX' is horizontal line called X-axis and YOY' is vertical line called Y-axis. The plane made by these axes is called Cartesian plane or coordinate plane.

The axes divide the plane into four parts called quadrant: 1st quadrant, 2nd quadrant, 3rd quadrant and 4th quadrant as shown in the figure. OX is known as positive direction of X-axis and OX' is known as negative direction of X-axis. X

Similarly, OY is known as positive direction of Y-axis and OY' is known as negative direction of Y-axis.

The axes XOX' and YOY' are together known as rectangular axes or coordinate axes.

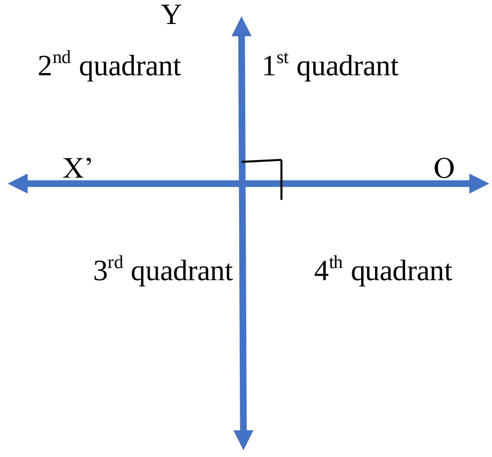


Fig. 3.1

Point: A point is a mark of location on a plane. It has no dimensions i.e. no length, no breadth and no height. For example, tip of pencil, toothpick etc. A point in a plane is represented as an ordered pair of real numbers called coordinates of point.

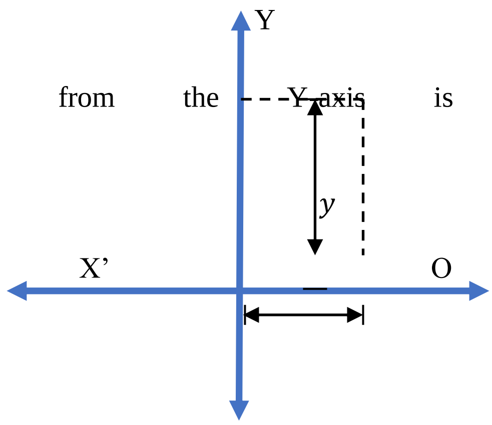
The perpendicular distance of a point from the Y-axis is $P(x, y)$

called abscissa or x-coordinate and the perpendicular

distance of a point from the X-axis is called ordinate or X

y-coordinate. If $P(x, y)$ be any point in the plane then xx

is the abscissa of the point P and y is the ordinate of the



point P.

Y'

Fig. 3.2

Note: **(i)** If distance along X-axis is measured to the right of Y-axis then it is positive and if it is measured to the left of Y-axis then it is negative.

(ii) If distance along Y-axis is measured to the above of X-axis then it is positive and if it is measured to the below of X-axis then it is negative.

(iii) The coordinates of origin 'O' are $(0,0)$.

(iv) A point on X-axis is represented as $(x, 0)$ i.e. ordinate is zero.

(v) A point on Y-axis is represented as $(0, y)$ i.e. abscissa is zero.

(vi) In 1st quadrant $x > 0$ and $y > 0$

In 2nd quadrant $x < 0$ and $y > 0$

In 3rd quadrant $x < 0$ and $y < 0$

In 4th quadrant $x > 0$ and $y < 0$.

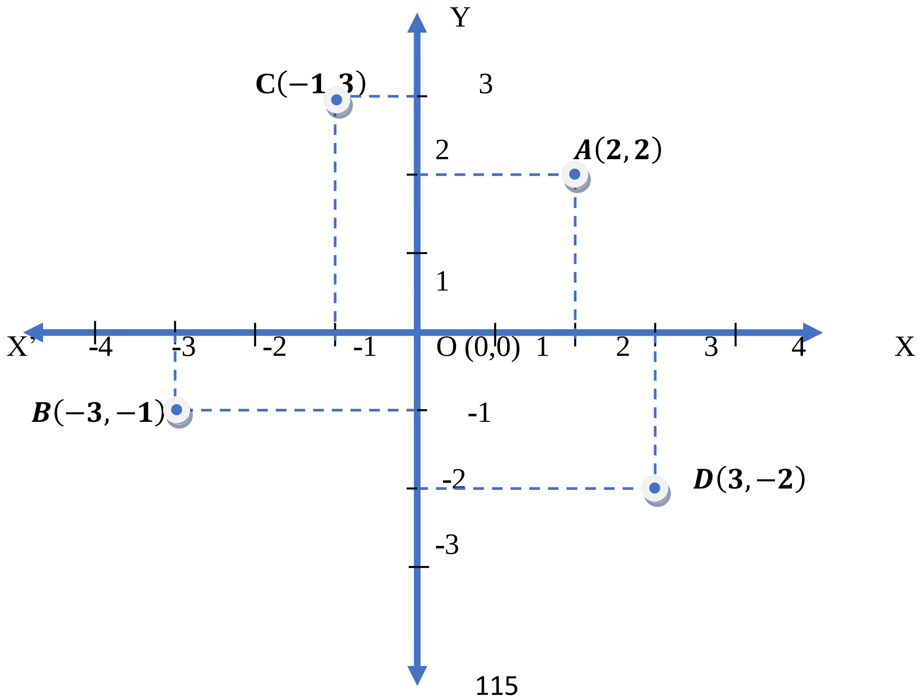
Distance between Two Points in a Plane: Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in a plane then the distance between them is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example1. Plot the following points and find the quadrant in which they lie:

(i) $\boldsymbol{A(2,2)}$ **(ii)** $\boldsymbol{B(-3,-1)}$ **(iii)** $\mathbf{C(-1,3)}$ **(iv)** $\boldsymbol{D(3,-2)}$

Sol.



Y'

Fig. 3.3

By graph it is clear that

(i) Point $\boldsymbol{A}(\mathbf{2},\mathbf{2})$ lies in the 1st quadrant.
(ii) Point $\boldsymbol{B}(-\mathbf{3},-\mathbf{1})$ lies in the 3rd quadrant.
(iii) Point $\boldsymbol{C}(-\mathbf{1},\mathbf{3})$ lies in the 2nd quadrant.
(iv) Point $\boldsymbol{D}(\mathbf{3},-\mathbf{2})$ lies in the 4th quadrant.

Example 2. Without plotting, find the quadrant in which the following points lie:

(i) $A(2,-3)$ (ii) $B(-5,-6)$ (iii) $\mathrm{C}(4,3)$ (iv) $D(-1,5)$

(v) $E(0,9)$ (vi) $F(-3,0)$ (vii) $\mathrm{G}(0,-7)$ (viii) $H(1,0)$

Sol.

(i) The given point is $A(2,-3)$

Here X-coordinate = 2, which is positive and Y-coordinate = -3, which is negative.

Hence the point $A(2,-3)$ lies in 4th quadrant.

(ii) The given point is $B(-5,-6)$

Here X-coordinate = -5, which is negative and Y-coordinate = -6, which is also negative.

Hence the point $B(-5,-6)$ lies in 3rd quadrant.

(iii) The given point is $\mathrm{C}(4,3)$

Here X-coordinate = 4 > 0 and Y-coordinate = 3 > 0.

Hence the point $\mathrm{C}(4,3)$ lies in 1st quadrant.

(iv) The given point is $D(-1,5)$

Here X-coordinate = -1 < 0 and Y-coordinate = 5 > 0.

Hence the point $D(-1,5)$ lies in 2nd quadrant.

(v) The given point is $E(0,9)$

Here X-coordinate = 0 and Y-coordinate = 9 > 0.

Hence the point $E(0,9)$ lies on Y-axis above the origin.

(vi) The given point is $F(-3,0)$

Here X-coordinate = -3 < 0 and Y-coordinate = 0.

Hence the point $F(-3,0)$ lies on X-axis left to origin.

(vii) The given point is $G(0,-7)$

Here X-coordinate = 0 and Y-coordinate = -7 < 0.

Hence the point $G(0,-7)$ lies on Y-axis below the origin.

(viii) The given point is $H(1,0)$

Here X-coordinate = 1 > 0 and Y-coordinate = 0.

Hence the point $H(1,0)$ lies on X-axis right to origin.

Example3. Find the distance between the following pairs of points:

(i) $(0,5)$, $(3,6)$ (ii) $(-1,2)$, $(4,3)$ (iii) $(2,0)$, $(-3,-2)$

(iv) $(1,2)$, $(4,5)$ (v) $(-2,3)$, $(-5,7)$ (vi) $(-1,-3)$, $(-2,-4)$

(vii) $(a-b, c-d)$, $(-b+c, c+d)$ (viii) $(sin\theta, cos\theta)$, $(-sin\theta, cos\theta)$

Sol.

(i) Let **A** represents the point $(0,5)$ and **B** represents the point $(3,6)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(3-0)^2+(6-5)^2}$$

$$=\sqrt{(3)^2+(1)^2}=\sqrt{9+1}=\sqrt{10}\,units$$

(ii) Let **A** represents the point $(-1,2)$ and **B** represents the point $(4,3)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(4-(-1))^2+(3-2)^2}$$

$$=\sqrt{(5)^2+(1)^2}=\sqrt{25+1}=\sqrt{26}\,units$$

(iii) Let **A** represents the point $(2,0)$ and **B** represents the point $(-3,-2)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(-3-2)^2+(-2-0)^2}$$

$$=\sqrt{(-5)^2+(-2)^2}=\sqrt{25+4}=\sqrt{29}\,units$$

(iv) Let **A** represents the point $(1,2)$ and **B** represents the point $(4,5)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(4-1)^2+(5-2)^2}$$

$$=\sqrt{(3)^2+(3)^2}=\sqrt{9+9}$$

$$=\sqrt{18}=\sqrt{9\times 2}=3\sqrt{2}\,units$$

(v) Let **A** represents the point $(-2,3)$ and **B** represents the point $(-5,7)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(-5-(-2))^2+(7-3)^2}$$

$$=\sqrt{(-5+2)^2+(4)^2}=\sqrt{(-3)^2+(4)^2}$$

$$=\sqrt{9+16}=\sqrt{25}=5units$$

(vi) Let **A** represents the point $(-1,-3)$ and **B** represents the point $(-2,-4)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(-2-(-1))^2+(-4-(-3))^2}$$

$$=\sqrt{(-2+1)^2+(-4+3)^2}=\sqrt{(-1)^2+(-1)^2}$$

$$=\sqrt{1+1}=\sqrt{2}\,units$$

(vii) Let **A** represents the point $(a-b, c-d)$ and **B** represents the point $(-b+c, c+d)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(-b+c-(a-b))^2+(c+d-(c-d))^2}$$

$$=\sqrt{(-b+c-a+b)^2+(c+d-c+d)^2}$$

$$=\sqrt{(c-a)^2+(d+d)^2}=\sqrt{c^2+a^2-2ac+(2d)^2}$$

$$=\sqrt{c^2+a^2+4d^2-2ac}\ \ units$$

(viii) Let **A** represents the point $(sin\theta, cos\theta)$ and **B** represents the point $(-sin\theta, cos\theta)$.

So, the distance between **A** and **B** is:

$$AB=\sqrt{(-\sin\theta-\sin\theta)^2+(\cos\theta-\cos\theta)^2}$$

$$=\sqrt{(-2\sin\theta)^2+(0)^2}=\sqrt{4\sin^2\theta}$$

$$=2\sin\theta\,units$$

Example 4. Using distance formula, prove that the triangle formed by the points $\boldsymbol{A(4,0)}$, $\boldsymbol{B(-1,-1)}$and $\boldsymbol{C(3,5)}$ is an isosceles triangle.

Sol. Given that vertices of the triangle are $A(4,0)$, $B(-1,-1)$ and $C(3,5)$.

To find the length of edges of the triangle, we will use the distance formula:

Distance between **A** and **B** is

$$AB=\sqrt{(4-(-1))^2+(0-(-1))^2}$$

$$=\sqrt{(5)^2+(1)^2}=\sqrt{25+1}=\sqrt{26}\,units$$

Distance between **B** and **C** is

$$BC=\sqrt{(-1-3)^2+(-1-5)^2}$$

$$=\sqrt{(-4)^2+(-6)^2}=\sqrt{16+36}=\sqrt{52}\,units$$

Distance between **A** and **C** is

$$AC=\sqrt{(4-3)^2+(0-5)^2}$$

$$=\sqrt{(1)^2+(-5)^2}=\sqrt{1+25}=\sqrt{26}\,units$$

We can see that $AB=AC\neq BC$

Hence the triangle formed by the points $A(4,0)$, $B(-1,-1)$ and $C(3,5)$ is an isosceles triangle.

Example 5. Using distance formula, prove that the triangle formed by the points $\boldsymbol{A(0,0)}$, $\boldsymbol{B(0,2)}$and $\boldsymbol{C(\sqrt{3},1)}$ is an equilateral triangle.

Sol. Given that vertices of the triangle are $A(0,0)$, $B(0,2)$ and $C(\sqrt{3},1)$.

To find the length of edges of the triangle, we will use the distance formula:

Distance between **A** and **B** is

$$AB=\sqrt{(0-0)^2+(0-2)^2}$$

$$=\sqrt{(0)^2+(-2)^2}=\sqrt{0+4}=\sqrt{4}\ =2\,units$$

Distance between **B** and **C** is

$$BC=\sqrt{\left(0-\sqrt{3}\right)^2+(2-1)^2}$$

$$=\sqrt{\left(-\sqrt{3}\right)^2+(1)^2}=\sqrt{3+1}=\sqrt{4}=2\,units$$

Distance between **A** and **C** is

$$AC=\sqrt{\left(0-\sqrt{3}\right)^2+(0-1)^2}$$

$$=\sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}=\sqrt{3+1}=\sqrt{4}=2\,units$$

We can see that $AB=BC=AC$

Hence the triangle formed by the points $A(0,0)$, $B(0,2)$ and $C(\sqrt{3},1)$ is an equilateral triangle.

Example 6. Find the mid points between the following pairs of points:

(i) $(2,3)$, $(8,5)$ (ii) $(6,3)$, $(6,-9)$ (iii) $(-2,-4)$, $(3,-6)$

(iv) $(0,8)$, $(6,0)$ (v) $(0,0)$, $(-12,10)$ (vi) (a,b), (c,d)

(vii) $(a+b,c-d)$, $(-b+3a,c+d)$

Sol.

(i) The given points are $(2,3)$ and $(8,5)$.

So, the mid-point between these points is given by:

$$\left(\frac{2+8}{2},\frac{3+5}{2}\right)=\left(\frac{10}{2},\frac{8}{2}\right)=(5,4)$$

(ii) The given points are $(6,3)$ and $(6,-9)$.

So, the mid-point between these points is given by:

$$\left(\frac{6+6}{2},\frac{3+(-9)}{2}\right)=\left(\frac{12}{2},\frac{3-9}{2}\right)=\left(\frac{12}{2},\frac{-6}{2}\right)=(6,-3)$$

(iii) The given points are $(-2,-4)$ and $(3,-6)$.

So, the mid-point between these points is given by:

$$\left(\frac{-2+3}{2},\frac{-4+(-6)}{2}\right)=\left(\frac{1}{2},\frac{-4-6}{2}\right)=\left(\frac{1}{2},\frac{-10}{2}\right)=\left(\frac{1}{2},-5\right)$$

(iv) The given points are $(0,8)$ and $(6,0)$.

So, the mid-point between these points is given by:

$$\left(\frac{0+6}{2},\frac{8+0}{2}\right)=\left(\frac{6}{2},\frac{8}{2}\right)=(3,4)$$

(v) The given points are $(0,0)$ and $(-12,10)$.

So, the mid-point between these points is given by:

$$\left(\frac{0+(-12)}{2},\frac{0+10}{2}\right)=\left(\frac{-12}{2},\frac{10}{2}\right)=(-6,5)$$

(vi) The given points are (a,b) and (c,d).

So, the mid-point between these points is given by:

$$\left(\frac{a+c}{2},\frac{b+d}{2}\right)$$

(vii) The given points are $(a+b,c-d)$and $(-b+3a,c+d)$.

So, the mid-point between these points is given by:

$$\left(\frac{a+b-b+3a}{2},\frac{c-d+c+d}{2}\right)=\left(\frac{4a}{2},\frac{2c}{2}\right)=(2a,c)$$

Example 7. If the mid-point between two points is $(3,5)$ and one point between them is $(-1,2)$, find the other point.

Sol. Let the required point is (a,b).

According to given statement $(3,5)$ is the mid-point of $(-1,2)$ and (a,b).

$$\Rightarrow \quad (3,5)=\left(\frac{-1+a}{2},\frac{2+b}{2}\right)$$

$$\Rightarrow \quad \frac{-1+a}{2}=3 \quad \& \quad \frac{2+b}{2}=5$$

$$\Rightarrow \quad -1+a=6 \quad \& \quad 2+b=10$$

$$\Rightarrow \quad a=7 \quad \& \quad b=8$$

Hence the required point is $(7,8)$.

Example8. If the mid-point between two points is $(-7,6)$ and one point between them is $(3,-9)$, find the other point.

Sol. Let the required point is (a,b).

According to given statement $(-7,6)$ is the mid-point of $(3,-9)$ and (a,b).

$$\Rightarrow \quad (-7,6)=\left(\frac{3+a}{2},\frac{-9+b}{2}\right)$$

$$\Rightarrow \qquad \frac{3+a}{2}=-7 \quad \& \quad \frac{-9+b}{2}=6$$

$$\Rightarrow \qquad 3+a=-14 \quad \& \quad -9+b=12$$

$$\Rightarrow \qquad a=-17 \quad \& \quad b=21$$

Hence the required point is $(-17,21)$.

<u>Centroid of a Triangle</u>: The centroid of a triangle is the intersection point of the three medians of the triangle. In other words, the **average** of the three vertices of the triangle is called the centroid of the triangle.

i.e. If $A(x_1,y_1)$,$B(x_2,y_2)$ and $C(x_3,y_3)$ are three vertices of a triangle then the centroid of the triangle is given by:

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$$

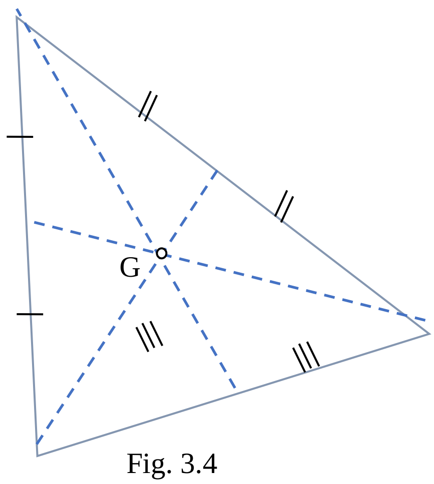


Fig. 3.4

In this fig.3.4, the point G is the centroid of the triangle.

Example9. Vertices of the triangles are given below, find the centroid of the triangles:

(i) $(5,2)$, $(5,4)$, $(8,6)$ (ii) $(4,-3)$, $(-4,8)$, $(5,7)$

(iii) $(2,-4)$, $(0,-10)$, $(4,5)$ (iv) $(9,-9)$, $(5,8)$, $(-7,-2)$

Sol.

(i) The given vertices of the triangle are $(5,2)$, $(5,4)$ and $(8,6)$.

So, the centroid of the triangle is

$$\left(\frac{5+5+8}{3},\frac{2+4+6}{3}\right)=\left(\frac{18}{3},\frac{12}{3}\right)=(6,4)$$

(ii) The given vertices of the triangle are $(4,-3)$, $(-4,8)$ and $(5,7)$.

So, the centroid of the triangle is

$$\left(\frac{4-4+5}{3},\frac{-3+8+7}{3}\right)=\left(\frac{5}{3},\frac{12}{3}\right)=\left(\frac{5}{3},4\right)$$

(iii) The given vertices of the triangle are $(2,-4)$, $(0,-10)$ and $(4,5)$.

So, the centroid of the triangle is

$$\left(\frac{2+0+4}{3},\frac{-4-10+5}{3}\right)=\left(\frac{6}{3},\frac{-9}{3}\right)=(2,-3)$$

(iv) The given vertices of the triangle are $(9,-9)$, $(5,8)$ and $(-7,-2)$.

So, the centroid of the triangle is

$$\left(\frac{9+5-7}{3},\frac{-9+8-2}{3}\right)=\left(\frac{7}{3},\frac{-3}{3}\right)=\left(\frac{7}{3},-1\right)$$

Example 10. If centroid of the triangle is $(10,18)$ and two vertices of the triangle are $(1,-5)$ and $(3,7)$, find the third vertex of the triangle.

Sol. Let the required vertex of the triangle is (a,b).

So, according to given statement and definition of centroid, we get

$$\Rightarrow \quad (10,18)=\left(\frac{1+3+a}{3},\frac{-5+7+b}{3}\right)$$

$$\Rightarrow \quad \frac{1+3+a}{3}=10 \quad \& \quad \frac{-5+7+b}{3}=18$$

$$\Rightarrow \quad 1+3+a=30 \quad \& \quad -5+7+b=54$$

$$\Rightarrow \quad a=26 \quad \& \quad b=52$$

Hence the required vertex of triangle is $(26,52)$.

Example 11. If centroid of the triangle is $(-5,-7)$ and two vertices of the triangle are $(0,6)$ and $(-3,2)$, find the third vertex of the triangle.

Sol. Let the required vertex of the triangle is (a,b).

So, according to given statement and definition of centroid, we get

$$\Rightarrow \quad (-5,-7)=\left(\frac{0-3+a}{3},\frac{6+2+b}{3}\right)$$

$$\Rightarrow \quad \frac{0-3+a}{3}=-5 \quad \& \quad \frac{6+2+b}{3}=-7$$

$$\Rightarrow \quad 0-3+a=-15 \quad \& \quad 6+2+b=-21$$

$$\Rightarrow \quad a=-12 \quad \& \quad b=-29$$

Hence the required vertex of triangle is $(-12,-29)$.

Example 12. If centroid of a triangle formed by the points $(\mathbf{1},\boldsymbol{a})$**,** $(\mathbf{9},\boldsymbol{b})$and $\left(\boldsymbol{c}^{\mathbf{2}},-\mathbf{5}\right)$ lies on the X-axis, prove that $\boldsymbol{a}+\boldsymbol{b}=\mathbf{5}$.

Sol. Given that vertices of the triangle are $(1, a)$, $(9, b)$ and $(c^2, -5)$.

Centroid of the triangle is given by

$$\left(\frac{1+9+c^2}{3}, \frac{a+b-5}{3}\right)$$

By given statement centroid lies on the X-axis. Therefore, Y-coordinate of centroid is zero

$$\Rightarrow \qquad \frac{a+b-5}{3}=0$$

$$\Rightarrow \qquad a+b-5=0$$

$$\Rightarrow \qquad a+b=5$$

Example 13. If centroid of a triangle formed by the points $(-\boldsymbol{a}, \boldsymbol{a})$, $(\boldsymbol{c}^2, \boldsymbol{b})$ and $(\boldsymbol{d}, \mathbf{5})$ lies on the Y-axis, prove that $\boldsymbol{c}^2 = \boldsymbol{a} - \boldsymbol{d}$.

Sol. Given that vertices of the triangle are $(-a, a)$, (c^2, b) and $(d, 5)$.

Centroid of the triangle is given by

$$\left(\frac{-a+c^2+d}{3}, \frac{a+b+5}{3}\right)$$

By given statement centroid lies on the Y-axis. Therefore, X-coordinate of centroid is zero

$$\Rightarrow \qquad \frac{-a+c^2+d}{3}=0$$

$$\Rightarrow \qquad -a+c^2+d=0$$

$$\Rightarrow \qquad c^2=a-d$$

Hence proved.

<u>Area of a Triangle with given vertices</u>:

If (x_1, y_1),(x_2, y_2) and (x_3, y_3) are vertices of a triangle then area of triangle is given by

$$\Delta = \pm\frac{1}{2}\left[(x_1y_2 - x_2y_1)+(x_2y_3 - x_3y_2)+(x_3y_1 - x_1y_3)\right]$$

To remember this we can take help of figure given below:

$$\Delta = \pm\frac{1}{2}\left|\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{array}\right|$$

Alternate Method: We can also find the area of triangle by the use of determinant if all the three vertices are given. If (x_1, y_1),(x_2, y_2) and (x_3, y_3) are vertices of a triangle then area of triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix}$$

Note: **(i)** Area is always non-negative. So take the suitable sign that gives the non-negative value.

(ii) If $\Delta = 0$ then the three points don't form triangle and these are **collinear** points.

Example 14. Vertices of the triangles are given below, find the area of the triangles:

(i) $(3,2)$, $(5,4)$, $(7,2)$ (ii) $(1,3)$, $(-4,5)$, $(3,-4)$

(iii) $(-2,1)$, $(2,-3)$, $(5,5)$ (iv) $(0,2)$, $(3,6)$, $(7,-5)$

Sol.

(i) The given vertices of the triangle are $(3,2)$, $(5,4)$ and $(7,2)$.

Comparing these points with (x_1, y_1),(x_2, y_2) and (x_3, y_3) respectively, we get

$x_1 = 3, y_1 = 2, x_2 = 5, y_2 = 4, x_3 = 7, y_3 = 2$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\left[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right]$$

i.e. $$\Delta = \pm\frac{1}{2}\left[(3\times 4 - 5\times 2) + (5\times 2 - 7\times 4) + (7\times 2 - 3\times 2)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(12-10) + (10-28) + (14-6)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[2-18+8\right] = \pm\frac{1}{2}\left[-8\right] = 4\,sq.units$$

Alternate Method

The given vertices of the triangle are $(3,2)$, $(5,4)$ and $(7,2)$.

Comparing these points with (x_1, y_1),(x_2, y_2) and (x_3, y_3) respectively, we get

$x_1 = 3, y_1 = 2, x_2 = 5, y_2 = 4, x_3 = 7, y_3 = 2$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 - x_2 & x_1 - x_3 \\ y_1 - y_2 & y_1 - y_3 \end{vmatrix}$$

i.e. $$\Delta = \pm\frac{1}{2}\begin{vmatrix} 3-5 & 3-7 \\ 2-4 & 2-2 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\begin{vmatrix} -2 & -4 \\ -2 & 0 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(-2)(0)-(-2)(-4)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[0-8\right] = 4\,sq.units$$

(ii) The given vertices of the triangle are $(1,3)$, $(-4,5)$ and $(3,-4)$.

Comparing these points with (x_1, y_1),(x_2, y_2) and (x_3, y_3) respectively, we get

$x_1 = 1, y_1 = 3, x_2 = -4, y_2 = 5, x_3 = 3, y_3 = -4$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\left[(x_1y_2 - x_2y_1)+(x_2y_3 - x_3y_2)+(x_3y_1 - x_1y_3)\right]$$

i.e. $$\Delta = \pm\frac{1}{2}\left[\{1\times 5-(-4)\times 3\}+\{-4\times(-4)-3\times 5\}+\{3\times 3-1\times(-4)\}\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(5+12)+(16-15)+(9+4)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[17+1+13\right] = \pm\frac{1}{2}\left[31\right] = 15.5\,sq.units$$

(iii) The given vertices of the triangle are $(-2,1)$, $(2,-3)$ and $(5,5)$.

Comparing these points with (x_1, y_1),(x_2, y_2) and (x_3, y_3) respectively, we get

$x_1 = -2, y_1 = 1, x_2 = 2, y_2 = -3, x_3 = 5, y_3 = 5$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} -2 & 1 \\ 2 & -3 \\ 5 & 5 \\ -2 & 1 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(6-2)+(10-(-15))+(5+10)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[4+25+15\right] = \pm\frac{1}{2}\left[44\right] = 22\, sq.units$$

(iv) The given vertices of the triangle are $(0,2)$, $(3,6)$ and $(7,-5)$.

Comparing these points with (x_1, y_1),(x_2, y_2) and (x_3, y_3) respectively, we get

$x_1 = 0, y_1 = 2, x_2 = 3, y_2 = 6, x_3 = 7, y_3 = -5$

So, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 0 & 2 \\ 3 & 6 \\ 7 & -5 \\ 0 & 2 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(0-6)+(-15-42)+(14-0)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[-6-57+14\right] = \pm\frac{1}{2}\left[-49\right] = 24.5\, sq.units$$

Example 15. If the area of the triangle with vertices $\mathbf{(1,1)}$, $\boldsymbol{(x,0)}$ **,** $\mathbf{(0,2)}$ is 4 sq. units, find the value of x.

Sol. The given vertices of the triangle are $(1,1)$, $(x, 0)$ and $(0, 2)$.

Comparing these points with (x_1, y_1),(x_2, y_2) and (x_3, y_3) respectively, we get

$x_1 = 1, y_1 = 1, x_2 = x, y_2 = 0, x_3 = 0, y_3 = 2$

Also it is given that the area of the triangle is 4 s**q.** units

i.e. $\Delta = 4\, sq.units$ (1)

Now, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 1 & 1 \\ x & 0 \\ 0 & 2 \\ 1 & 1 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(0-x)+(2x-0)+(0-2)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[-x+2x-2\right]=\pm\frac{1}{2}\left[x-2\right] sq.units \qquad (2)$$

Comparing (1) and (2), we get

$$\pm\frac{1}{2}\left[x-2\right]=4$$

$\Rightarrow$ $$\pm\left[x-2\right]=8$$

$\Rightarrow$ $\quad$ *either* $x-2=8$ *or* $x-2=-8$

$\Rightarrow$ $\quad$ *either* $x=10$ *or* $x=-6$

which is the required sol.

Example 16. Prove that the triplet of points $(\mathbf{4},\mathbf{7})$, $(\mathbf{0},\mathbf{1})$ **,** $(\mathbf{2},\mathbf{4})$ is collinear.

Sol. The points are $(4,7)$, $(0,1)$ and $(2,4)$.

Comparing these points with (x_1, y_1),(x_2, y_2) and (x_3, y_3) respectively, we get

$x_1 = 4, y_1 = 7$,$x_2 = 0, y_2 = 1$,$x_3 = 2, y_3 = 4$

So by formula of area of the triangle, we get

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 4 & 7 \\ 0 & 1 \\ 2 & 4 \\ 4 & 7 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(4-0)+(0-2)+(14-16)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[4-2-2\right]=0$$

which shows that the given points are collinear.

Example 17. Find the value of x, in order that the points $(\mathbf{5},-\mathbf{1})$, $(-\mathbf{4},\mathbf{2})$ **and** $(\boldsymbol{x},\mathbf{6})$ are collinear.

Sol. The given points are $(5,-1)$, $(-4,2)$ and $(x,6)$.

Comparing these points with (x_1,y_1),(x_2,y_2) and (x_3,y_3) respectively, we get

$x_1 = 5, y_1 = -1, x_2 = -4, y_2 = 2, x_3 = x, y_3 = 6$

Also it is given that the points $(5,-1)$, $(-4,2)$ and $(x,6)$ are collinear

i.e. $\Delta=0$ (1)

Now, area of the triangle is given by

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

i.e.

$$\Delta = \pm\frac{1}{2}\begin{vmatrix} 5 & -1 \\ -4 & 2 \\ x & 6 \\ 5 & -1 \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[(10-4)+(-24-2x)+(-x-30)\right]$$

$\Rightarrow$ $$\Delta = \pm\frac{1}{2}\left[6-24-2x-x-30\right]=\pm\frac{1}{2}\left[-3x-48\right] sq.units \quad (2)$$

Comparing (1) and (2), we get

$$\pm\frac{1}{2}\left[-3x-48\right]=0$$

$\Rightarrow \qquad -3x-48=0$

$\Rightarrow \qquad -3x=48$

$\Rightarrow \qquad x=-\dfrac{48}{3}$

$\Rightarrow \qquad x=-16$

which is the required solution.

EXERCISE – I

1. The point (-3, -4) lies in quadrant:

(a) First (b) Second (c) Third (d) Fourth

2. Three points are collinear and Δ is the area of triangle formed with these three points, then

(a) $\Delta = 0$ (b) $\Delta > 0$ (c) $\Delta < 0$ (d) 1

3. Find the distance between the following pairs of points:

(i) (-1, 2), (4, 3) (ii) (a - b, c - d), (-b + c, c + d)

4. Find the mid points between the following pairs of points:

(i) (0, 8), (6, 0) (ii) (a + b, c - d), (-b + 3a, c + d)

5. The midpoint between two points is (3, 5) and one point between them is (-1, 2). Find the other point.

6. Find the centroid of the triangle whose vertices are:

(i) (4, -3), (-4, 8), (5, 7) (ii) (9, -9), (5, 8), (-7, -2)

7. Find the area of the triangles whose vertices are:

(i) (1, 3), (-4, 5), (3, -4) (ii) (0, 2), (3, 6), (7, -5)

8. Prove that the triplet of points (4, 7), (0, 1), (2, 4) is collinear.

ANSWERS

1. (d) 2. (a) 3. (i) [$\sqrt{26}$] (ii) [$\sqrt{c^2 + a^2 + 4d^2 - 2ac}$] 4. (i) (3,4) (ii) (2a, c)

5. (7, 8) 6. (i) (5/3, 4) (ii) (7/3, -1) 7. (i) 15.5 (ii) 24.5

3.2 STRAIGHT LINE

Definition: A path traced by a point travelling in a constant direction is called a straight line.

OR

The shortest path between two points is called a straight line.

General Equation of Straight Line: A straight line in XY plane has general form

$$ax + by + c = 0$$

where a is the coefficient of x, b is the coefficient of y and c is the constant term so that at least one of a,b is non-zero.

Note: **(i)** Any point (x_1, y_1) lies on the line $ax + by + c = 0$ if it satisfies the equations of the line *i.e.* if we substitute the values x_1 at the place of x and y_1 at the place of y in the equation of line, the result $ax_1 + by_1 + c$ becomes zero.

(ii) X-axis is usually represented horizontally and its equation is $y = 0$.

(iii) Y-axis is usually represented vertically and its equation is $x = 0$.

(iv)$x = k$ represents the line parallel to Y-axis, where k is some constant .

(v)$y = k$ represents the line parallel to X-axis, where k is some constant .

Slope of a Straight Line: Slope of straight line measures with tangent of the angle of straight line to the horizon.

It is usually represented by $\boldsymbol{m}$.

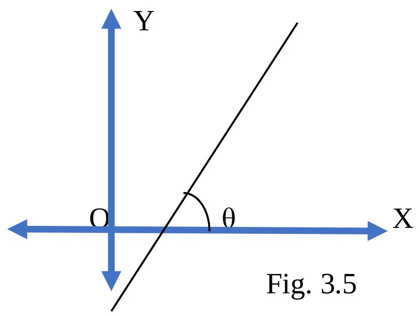


Fig. 3.5

To find slope of a straight Line:

(i) If a non-vertical line making an angle θ with positive X-axis then the slope m of the line is given by $m = tan\theta$.

(ii) If a non-vertical line passes through two points (x_1, y_1) and (x_2, y_2) then the slope m of the line is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

(iii) If equation of a straight line is $ax + by + c = 0$, then its slope m is given by $m = -\dfrac{a}{b}$.

<u>Note</u>: **(i)** Slop of a horizontal line is always zero **i.e.,** slope of a line parallel to X-axis is zero as $m = tan0° = 0$.

(ii) Slop of a vertical line is always infinity **i.e.,** slope of a line perpendicular to X-axis is infinity as $= tan90° = \infty$.

(iii) Let L_1 and L_2 represents two straight lines. Let m_1 and m_2 be slopes of L_1 and L_2 respectively. We say that L_1 and L_2 are parallel lines iff $m_1 = m_2$**i.e.** slopes are equal. We say that L_1 and L_2 are perpendicular iff $m_1 . m_2 = -1$**i.e.,** product of slopes is equal to -1.

Example 18. Find the slope of the straight lines which make following angles:

(i) 45° (ii) 120° (iii) 30° (iv) 150° (v) 210°

with the positive direction of X-axis.

Sol.

(i) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.
Therefore $\theta = 45°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 45^\circ$

$\Rightarrow \quad m = 1$

which is the required slope.

(ii) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.
Therefore $\theta = 120°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 120^\circ$

$\Rightarrow \quad m = \tan\left(180^\circ - 60^\circ\right)$

$\Rightarrow \quad m = -\tan\left(60^\circ\right)$

$\Rightarrow \quad m = -\sqrt{3}$

which is the required slope.

(iii) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.
Therefore $\theta = 30°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 30^\circ$

$\Rightarrow \quad m = \dfrac{1}{\sqrt{3}}$

which is the required slope.

(iv) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.
Therefore $\theta = 150°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 150^\circ$

$\Rightarrow \quad m = \tan\left(180^\circ - 30^\circ\right)$

$\Rightarrow \quad m = -\tan(30^\circ)$

$\Rightarrow \quad m = -\frac{1}{\sqrt{3}}$

which is the required slope.

(v) Let m be the slope of the straight line and θ be the angle which the straight line makes with the positive direction of X-axis.

Therefore $\theta = 210°$ and $m = tan\theta$

$\Rightarrow \quad m = \tan 210^\circ$

$\Rightarrow \quad m = \tan(180^\circ + 30^\circ)$

$\Rightarrow \quad m = \tan(30^\circ)$

$\Rightarrow \quad m = \frac{1}{\sqrt{3}}$

which is the required slope.

Example 19. Find the slope of the straight lines which pass through the following pairs of points:

(i) $(2,5)$, $(6,17)$ (ii) $(-8,7)$, $(3,-5)$ (iii) $(0,-6)$, $(7,9)$

(iv) $(-11,-5)$, $(-3,-10)$ (v) $(0,0)$, $(10,-12)$.

Sol.

(i) Given that the straight line passes through the points $(2,5)$ and $(6,17)$.

Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get

$x_1 = 2$, $y_1 = 5$, $x_2 = 6$ and $y_2 = 17$

Let m be the slope of the straight line.

Therefore $m = \frac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \frac{17-5}{6-2} = \frac{12}{4}$

$\Rightarrow \quad m = 3$

which is the required slope.

(ii) Given that the straight line passes through the points $(-8,7)$ and $(3,-5)$.

Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get

$x_1 = -8$, $y_1 = 7$, $x_2 = 3$ and $y_2 = -5$

Let m be the slope of the straight line.

Therefore $m = \frac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \frac{-5-7}{3-(-8)} = \frac{-12}{3+8}$

$\Rightarrow \quad m = -\frac{12}{11}$

which is the required slope.

(iii) Given that the straight line passes through the points $(0,-6)$ and $(7,9)$.

Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get
$x_1 = 0$, $y_1 = -6$, $x_2 = 7$ and $y_2 = 9$
Let m be the slope of the straight line.

Therefore $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \dfrac{9 - (-6)}{7 - 0} = \dfrac{9 + 6}{7}$

$\Rightarrow \quad m = \dfrac{15}{7}$

which is the required slope.

(iv) Given that the straight line passes through the points $(-11, -5)$ and $(-3, -10)$.
Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get
$x_1 = -11$, $y_1 = -5$, $x_2 = -3$ and $y_2 = -10$
Let m be the slope of the straight line.

Therefore $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \dfrac{-10 - (-5)}{-3 - (-11)} = \dfrac{-10 + 5}{-3 + 11}$

$\Rightarrow \quad m = -\dfrac{5}{8}$

which is the required slope.

(v) Given that the straight line passes through the points $(0,0)$ and $(10, -12)$.
Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get
$x_1 = 0$, $y_1 = 0$, $x_2 = 10$ and $y_2 = -12$
Let m be the slope of the straight line.

Therefore $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \quad m = \dfrac{-12 - 0}{10 - 0} = \dfrac{-12}{10}$

$\Rightarrow \quad m = -\dfrac{6}{5}$

which is the required slope.

Example 20. Find the slopes of the following straight lines:

(i) $2x + 4y + 5 = 0$ (ii) $x - 3y + 9 = 0$

(iii) $5y - 10x + 1 = 0$ (iv) $-2x - 6y = 0$

(v) $x = 5$ (vi) $y = -6$

Sol.

(i) Given that equation of the straight line is $2x + 4y + 5 = 0$.
Comparing this equation with $ax + by + c = 0$, we get
$a = 2$, $b = 4$ and $c = 5$

Let m be the slope of given straight line.

Therefore, $m = -\frac{a}{b}$

$\Rightarrow \quad m = -\frac{2}{4}$

$\Rightarrow \quad m = -\frac{1}{2}$

which is the required slope.

(ii) Given that equation of the straight line is $-3y + 9 = 0$.
Comparing this equation with $ax + by + c = 0$, we get
$a = 1$, $b = -3$ and $c = 9$
Let m be the slope of given straight line.

Therefore, $m = -\left(\frac{1}{-3}\right)$

$\Rightarrow \quad m = \frac{1}{3}$

which is the required slope.

(iii) Given that equation of the straight line is $5y - 10x + 1 = 0$.
Comparing this equation with $ax + by + c = 0$, we get
$a = -10$, $b = 5$ and $c = 1$
Let m be the slope of given straight line.

Therefore, $m = -\left(\frac{-10}{5}\right)$

$\Rightarrow \quad m = 2$

which is the required slope.

(iv) Given that equation of the straight line is $-2x - 6y = 0$.
Comparing this equation with $ax + by + c = 0$, we get
$a = -2$, $b = -6$ and $c = 0$
Let m be the slope of given straight line.

Therefore, $m = -\frac{a}{b}$

$\Rightarrow \quad m = -\left(\frac{-2}{-6}\right)$

$\Rightarrow \quad m = -\frac{1}{3}$

which is the required slope.

(v) Given that equation of the straight line is $= 5$.
This equation is parallel to Y-axis.
Hence the slope of the line is infinity.

(vi) Given that equation of the straight line is $= -6$.
This equation is parallel to X-axis.
Hence the slope of the line is zero.

Example 21. Find the equation of straight line which is parallel to X-axis passes through $(1,5)$.

Sol. Equation of straight line parallel to X-axis is given by

$$y = k$$

Given that the straight line passes through the point $(1,5)$.

Put $x = 1$ and $y = 5$ in (1), we get

$$5 = k$$

So, $y = 5$ be the required equation of straight line.

Example 22. Find the equation of straight line which is parallel to Y-axis passes through $(-3, -7)$.

Sol. Equation of straight line parallel to Y-axis is given by

$$x = k$$

Given that the straight line passes through the point $(-3, -7)$.

Put $x = -3$ and $y = -7$ in (1), we get

$$-3 = k$$

So, $x = -3$ be the required equation of straight line.

Equation of Straight Line Passing Through Origin:

If a non-vertical line passes through origin and m be its slope. $P(x, y)$ be any point on the line (see Fig. 3.6), then equation of straight line is $\boldsymbol{y = mx}$

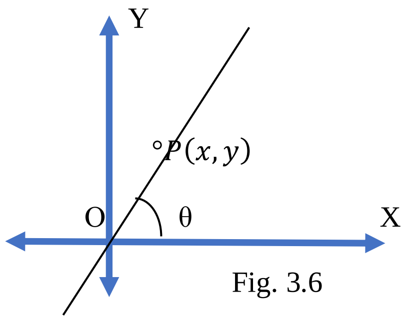


Fig. 3.6

Example 23. Find the equation of straight line having slope equal to 5 and passes through origin.

Sol. Let m be the slope of required line. Therefore $m = 5$.

Also it is given that the required line passes through the origin.We know that equation of straight line passes through origin is $y = mx$, where m be the slope of the line.

So, $y = 5x$ be the required equation of straight line.

Example 24 Find the equation of straight line having slope equal to -10 and passes through origin.

Sol. Let m be the slope of required line. Therefore $m = -10$. Also it is given that the required line passes through the origin.

We know that equation of straight line passes through origin is $y = mx$, where m be the slope of the line. So, $y = -10x$ be the required equation of straight line.

Example 25. Find the equation of straight line which passes through origin and makes an angle 60° with the positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 60^\circ$

$\Rightarrow \quad m = \sqrt{3}$

Also it is given that the required line passes through the origin. We know that equation of straight line passes through origin is $y = mx$, where m be the slope of the line.

So, $y = \sqrt{3}\,x$ be the required equation of straight line.

Example 26. Find the equation of straight line which passes through origin and makes an angle 135° with the positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 135^\circ$

$\Rightarrow \quad m = \tan(180^\circ - 45^\circ)$

$\Rightarrow \quad m = -\tan 45^\circ$

$\Rightarrow \quad m = -1$

Also it is given that the required line passes through the origin.We know that equation of straight line passes through origin is $y = mx$, where m be the slope of the line.

So, $y = -x$ be the required equation of straight line.

<u>Equation of Straight Line in Point-Slope form</u>:

Let a non-vertical line passes through a point (x_1, y_1) and m be its slope. $P(x, y)$ be any point on the line (see fig. 3.7), then equationof straight line is $\boldsymbol{y - y_1 = m(x - x_1)}$

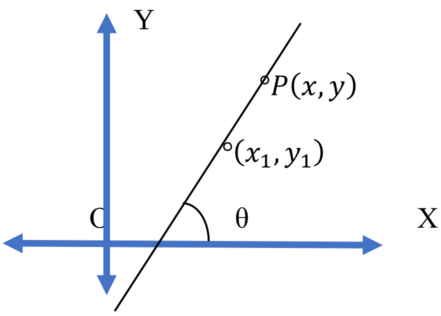


Fig. 3.7

Example 27. Find the equation of straight line having slope equal to 9 and passes through the point $(1,5)$.

Sol. Let m be the slope of required line. Therefore $m = 9$.

Also it is given that the required line passes through the point $(1,5)$. We know that equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

$\Rightarrow \quad y-5=9(x-1)$

$\Rightarrow \quad y-5=9x-9$

$\Rightarrow \quad 9x-y-9+5=0$

$\Rightarrow \quad 9x-y-4=0$

which is the required equation of straight line.

Example 28. Find the equation of straight line passes through $(-4,-2)$ and having slope -8.

Sol. Let m be the slope of required line. Therefore $m = -8$. Also it is given that the required line passes through the point $(-4,-2)$. We know that equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

$\Rightarrow \quad y-(-2)=-8(x-(-4))$

$\Rightarrow \quad y+2=-8(x+4)$

$\Rightarrow \quad y+2=-8x-32$

$\Rightarrow \quad 8x+y+2+32=0$

$\Rightarrow \quad 8x+y+34=0$

which is the required equation of straight line.

Example 29. Find the equation of straight line passes through $(0,-8)$ and makes an angle $30°$ with positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 30^\circ$

$$\Rightarrow \quad m = \frac{1}{\sqrt{3}}$$

Also it is given that the required line passes through the point $(0,-8)$.

We know that equation of straight line in point slope form is $y - y_1 = m(x - x_1)$.

$$\Rightarrow \quad y-(-8)=\frac{1}{\sqrt{3}}(x-0)$$

$$\Rightarrow \quad y+8=\frac{x}{\sqrt{3}}$$

$$\Rightarrow \quad \sqrt{3}\,y+8\sqrt{3}=x$$

$$\Rightarrow \quad x-\sqrt{3}\,y-8\sqrt{3}=0$$

which is the required equation of straight line.

Example 30. Find the equation of straight line passes through $(-9,0)$ and makes an angle $150°$ with positive direction of X-axis.

Sol. Let m be the slope of required line.

Therefore $m = \tan 150^\circ$

$$\Rightarrow \quad m=\tan\left(180^\circ-30^\circ\right)$$

$$\Rightarrow \quad m=-\tan\left(30^\circ\right)$$

$$\Rightarrow \quad m=-\frac{1}{\sqrt{3}}$$

Also it is given that the required line passes through the point $(-9,0)$. We know that equation of straight line in point slope form is $y-y_1=m\left(x-x_1\right)$.

$$\Rightarrow \quad y-0=-\frac{1}{\sqrt{3}}\left(x-(-9)\right)$$

$$\Rightarrow \quad -\sqrt{3}\,y=x+9$$

$$\Rightarrow \quad x+\sqrt{3}\,y+9=0$$

which is the required equation of straight line.

Equation of Straight Line in Two Points form:

Let a non-vertical line passes through two points (x_1,y_1) and (x_2,y_2). $P(x,y)$ be any point on the line (see fig. 3.8), then equation of straight line is $\boldsymbol{y-y_1=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x-x_1)}$

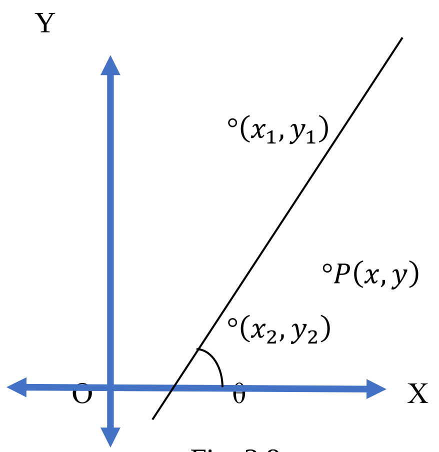


Fig. 3.8

Example 31. Find the equation of straight line passes through the points $(2,-2)$ and $(0,6)$.

Sol. Given that the straight line passes through the points $(2,-2)$ and $(0,6)$.

Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get

$x_1 = 2$, $y_1 = -2$, $x_2 = 0$ and $y_2 = 6$. We know that equation of straight line in two points slope form is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

$$\Rightarrow \quad y - (-2) = \left(\frac{6 - (-2)}{0 - 2}\right)(x - 2)$$

$$\Rightarrow \quad y + 2 = \left(\frac{6 + 2}{-2}\right)(x - 2)$$

$$\Rightarrow \quad y + 2 = -4(x - 2)$$

$$\Rightarrow \quad y + 2 = -4x + 8$$

$$\Rightarrow \quad 4x + y - 6 = 0$$

which is the required equation of straight line.

Example 32. Find the equation of straight line passes through the points $(0,8)$ and $(5,0)$.

Sol. Given that the straight line passes through the points $(0,8)$ and $(5,0)$.

Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get

$x_1 = 0$, $y_1 = 8$, $x_2 = 5$ and $y_2 = 0$. We know that equation of straight line in two points slope form is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

$$\Rightarrow \quad y - 8 = \left(\frac{0 - 8}{5 - 0}\right)(x - 0)$$

$$\Rightarrow \quad y - 8 = \frac{-8x}{5}$$

$$\Rightarrow \quad 5y - 40 = -8x$$

$$\Rightarrow \quad 8x + 5y - 40 = 0$$

which is the required equation of straight line.

Example 33. Find the equation of straight line passes through the points $(7, -4)$ and $(-1,5)$.

Sol. Given that the straight line passes through the points $(7, -4)$ and $(-1,5)$.

Comparing these points with (x_1, y_1) and (x_2, y_2) respectively, we get

$x_1 = 7$, $y_1 = -4$, $x_2 = -1$ and $y_2 = 5$. We know that equation of straight line in two points slope form is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

$$\Rightarrow \quad y - (-4) = \left(\frac{5 - (-4)}{-1 - 7}\right)(x - 7)$$

$$\Rightarrow \quad y+4=-\frac{9}{8}(x-7)$$
$$\Rightarrow \quad 8y+32=-9x+63$$
$$\Rightarrow \quad 9x+8y-31=0$$
which is the required equation of straight line.

Equation of Straight Line in Slope-Intercept form:

Let a non-vertical line having slope m and its y-intercept is equal to c. $P(x, y)$ be any point on the line (see Fig. 3.9), then equation of straight line is $\boldsymbol{y = mx + c}$

X

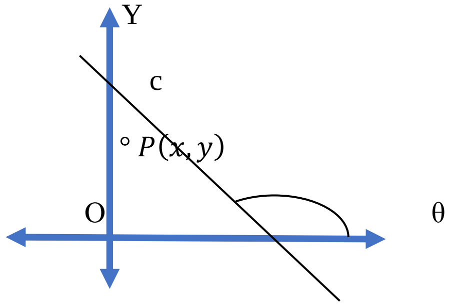


Fig. 3.9

Note: **(i)** If intercept c is given above the X-axis or above the origin then it is positive.

(ii) If intercept c is given below the X-axis or below the origin then it is negative.

Example 34. Find the equation of straight line having slope 3 and cuts of an intercept -2 on Y-axis.

Sol. Given that the slope m of straight line is 3 and Y-intercept is -2 i.e. $c = -2$. We know that equation of straight line in slope-intercept form is

$$y=mx+c$$
$$\Rightarrow \quad y=3x-2$$
$$\Rightarrow \quad 3x-y-2=0$$
which is the required equation of straight line.

Example 35. Find the equation of straight line having slope -6 and cuts of an intercept 5 on Y-axis above the origin.

Sol. Given that the slope m of straight line is -6 and Y-intercept is 5 i.e. $c = 5$.

c is taken positive as Y-intercept is above the origin.

We know that equation of straight line in slope-intercept form is

$$y=mx+c$$
$$\Rightarrow \quad y=-6x+5$$
$$\Rightarrow \quad 6x+y-5=0$$

which is the required equation of straight line.

Example 36. Find the equation of straight line having slope 2 and cuts of an intercept 9 on Y-axis below the origin.

Sol. Given that the slope m of straight line is 2 and Y-intercept is -9 i.e. $c = -9$. c is taken negative as Y-intercept is below the origin. We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$$\Rightarrow \quad y = 2x - 9$$

$$\Rightarrow \quad 2x - y - 9 = 0$$

which is the required equation of straight line.

Example 37. Find the equation of straight line which makes an angle 45° with X-axis and cuts of an intercept 8 on Y-axis below the X-axis.

Sol. Given that the required line makes an angle 45° with X-axis.

Therefore slope m of straight line is given by $m = \tan 45^\circ$ *i.e.* $m = 1$.

Also Y-intercept is -8 i.e. $c = -8$. c is taken negative as Y-intercept is below the X-axis. We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$$\Rightarrow \quad y = 1x - 8$$

$$\Rightarrow \quad x - y - 8 = 0$$

which is the required equation of straight line.

Example 38. Find the equation of straight line which makes an angle 60° with X-axis and cuts of an intercept 5 on Y-axis above the X-axis.

Sol. Given that the required line makes an angle 60° with X-axis.

Therefore slope m of straight line is given by $m = \tan 60^\circ$ *i.e.* $m = \sqrt{3}$.

Also Y-intercept is 5 i.e. $c = 5$. c is taken positive as Y-intercept is above the X-axis. We know that equation of straight line in slope-intercept form is

$$y = mx + c$$

$$\Rightarrow \quad y = \sqrt{3}x + 5$$

$$\Rightarrow \quad \sqrt{3}x - y + 5 = 0$$

which is the required equation of straight line.

Example 39. Find the equation of straight line which passes through the points $(0,3)$ and $(2,0)$ and cuts of an intercept 12 on Y-axis below the origin.

Sol. Given that the required line passes through the points $(0,3)$ and $(2,0)$. Therefore slope m of straight line is given by $m=\frac{0-3}{2-0}$ *i.e.* $m=-\frac{3}{2}$. Also Y-intercept is -12 i.e. $c=-12$. c is taken negative as Y-intercept is below the origin. We know that equation of straight line in slope-intercept form is

$$y=mx+c$$

$$\Rightarrow \quad y=-\frac{3}{2}x-12$$

$$\Rightarrow \quad 2y=-3x-24$$

$$\Rightarrow \quad 3x+2y+24=0$$

which is the required equation of straight line.

<u>Equation of Straight Line in Intercept form</u>:

Let a non-vertical line having intercepts a and b on X-axis and Y-axis respectively. $P(x,y)$ be any point on the line (Fig. 3.10), then equation of straight line is

$$\boldsymbol{\frac{x}{a}+\frac{y}{b}=1}$$

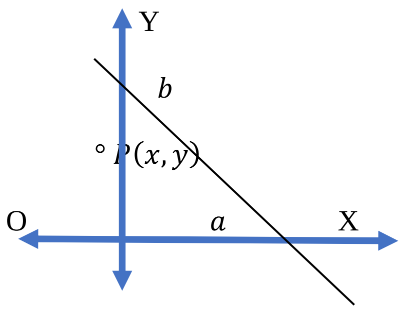


Fig. 3.10

Example 40. Find the equation of straight line which makes intercepts 2 and 5 on X-axis and Y-axis respectively.

Sol. Given that X-intercept is 2 and Y-intercept is 5

i.e. $a=2$ and $b=5$

We know that equation of straight line in Intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \quad \frac{x}{2}+\frac{y}{5}=1$$

$$\Rightarrow \quad \frac{5x+2y}{10}=1$$

$$\Rightarrow \quad 5x+2y=10$$

$$\Rightarrow \quad 5x+2y-10=0$$

which is the required equation of straight line.

Example 41. Find the equation of straight line which makes intercepts 3 and -15 on the axes.

Sol. Given that X-intercept is 3 and Y-intercept is -15

i.e. $a = 3$ and $b = -15$

We know that equation of straight line in Intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \quad \frac{x}{3}+\frac{y}{-15}=1$$

$$\Rightarrow \quad \frac{x}{3}-\frac{y}{15}=1$$

$$\Rightarrow \quad \frac{5x-y}{15}=1$$

$$\Rightarrow \quad 5x-y=15$$

$$\Rightarrow \quad 5x-y-15=0$$

which is the required equation of straight line.

Example 42. Find the equation of straight line which passes through $(1,-4)$ and makes intercepts on axes which are equal in magnitude and opposite in sign.

Sol. Let the intercepts on the axes are p and $-p$

i.e. $a = p$ and $b = -p$

We know that equation of straight line in Intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \quad \frac{x}{p}+\frac{y}{-p}=1$$

$$\Rightarrow \quad \frac{x}{p}-\frac{y}{p}=1$$

$$\Rightarrow \quad \frac{x-y}{p}=1$$

$$\Rightarrow \quad x-y=p \qquad (1)$$

Given that this line passes through $(1,-4)$.

Therefore put $x = 1$ and $y = -4$ in (1), we get

$$1-(-4)=p$$

$$\Rightarrow \quad p=5$$

Using this value in (1), we get

$$x-y=5$$

$$\Rightarrow \quad x-y-5=0$$

which is the required equation of straight line.

Example 43. Find the equation of straight line which passes through $(1,4)$ and sum of whose intercepts on axes is 10.

Sol. Let the intercepts on the axes are p and $10-p$

$i.e. a = p$ and $b = 10 - p$

We know that equation of straight line in Intercept form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\Rightarrow \quad \frac{x}{p}+\frac{y}{10-p}=1$$

$$\Rightarrow \quad \frac{(10-p)x+py}{p(10-p)}=1$$

$$\Rightarrow \quad (10-p)x+py=p(10-p) \qquad (1)$$

Given that this line passes through $(1,4)$. Therefore put $x=1$ and $y=4$ in (1), we get

$$(10-p)(1)+p(4)=p(10-p)$$

$$\Rightarrow \quad 10-p+4p=10p-p^2$$

$$\Rightarrow \quad p^2-7p+10=0$$

$$\Rightarrow \quad p^2-5p-2p+10=0$$

$$\Rightarrow \quad p(p-5)-2(p-5)=0$$

$$\Rightarrow \quad (p-2)(p-5)=0$$

either $p-2=0$ *or* $p-5=0$

either $p=2$ *or* $p=5$

Put $p=2$ in (1), we get

$$\Rightarrow \quad (10-2)x+2y=2(10-2)$$

$$\Rightarrow \quad 8x+2y=16$$

$$\Rightarrow \quad 4x+y=8 \qquad (2)$$

Put $p=5$ in (1), we get

$$\Rightarrow \quad (10-5)x+5y=5(10-5)$$

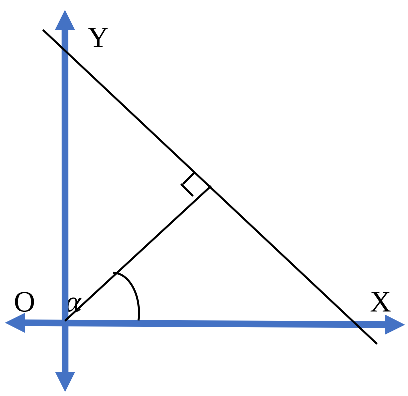
$$\Rightarrow \quad 5x+5y=25$$

$$\Rightarrow \quad x+y=5 \qquad (3)$$

(2) and (3) are the required equations of straight lines.

<u>Equation of Straight Line in Normal form</u>:

Let p be the length of perpendicular from the origin to the straight line and α be the angle which this perpendicular makes with the positive direction of X-axis. (x, y) be any point on the line (see Fig.p 3.11), then equation of straight line is



$$p = x\,cos\alpha + y\,sin\alpha$$

Fig. 3.11

Example 44. Find the equation of straight line such that the length of perpendicular from the origin to the straight line is 2 and the inclination of this perpendicular to the X-axis is $120°$.

Sol. We know that equation of straight line in Normal form is

$$x\cos\alpha + y\sin\alpha = p \qquad (1)$$

where p be the length of perpendicular from the origin to the straight line and α be the angle which this perpendicular makes with the positive direction of X-axis.

Here $p=2$ and $\alpha = 120^\circ$. Putting these values in (1), we get

$$x\cos 120^\circ + y\sin 120^\circ = 2$$

$$\Rightarrow \quad x\cos(180^\circ - 60^\circ) + y\sin(180^\circ - 60^\circ) = 2$$

$$\Rightarrow \quad -x\cos(60^\circ) + y\sin(60^\circ) = 2$$

$$\Rightarrow \quad -x\left(\frac{1}{2}\right) + y\left(\frac{\sqrt{3}}{2}\right) = 2$$

$$\Rightarrow \quad \frac{-x+\sqrt{3}\,y}{2} = 2$$

$$\Rightarrow \quad -x+\sqrt{3}\,y = 4$$

$$\Rightarrow \quad -x+\sqrt{3}\,y - 4 = 0$$

which is the required equation of straight line.

Example 45. Find the equation of straight line such that the length of perpendicular from the origin to the straight line is 7 and the inclination of this perpendicular to the X-axis is $45°$.

Sol. We know that equation of straight line in Normal form is

$$x\cos\alpha + y\sin\alpha = p \qquad (1)$$

where p be the length of perpendicular from the origin to the straight line and α be the angle which this perpendicular makes with the positive direction of X-axis.

Here $p=7$ and $\alpha = 45^\circ$. Put these values in (1), we get

$$x\cos 45^\circ + y\sin 45^\circ = 7$$

$$\Rightarrow \quad x\left(\frac{1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) = 7$$

$$\Rightarrow \quad x + y = 7\sqrt{2}$$

$$\Rightarrow \qquad x + y - 7\sqrt{2} = 0$$

which is the required equation of straight line.

Angle Between Two Straight Lines:

Two intersecting lines always intersects at two angles in which one angle is acute angle and other angle is obtuse angle. The sum of both the angles is 180° i.e. they are supplementary to each other. For Ex, if one angle between intersecting lines is 60° then other angle is $180^\circ - 60^\circ = 120^\circ$. Generally, we take acute angle as the angle between the lines (see Fig. 3.12).

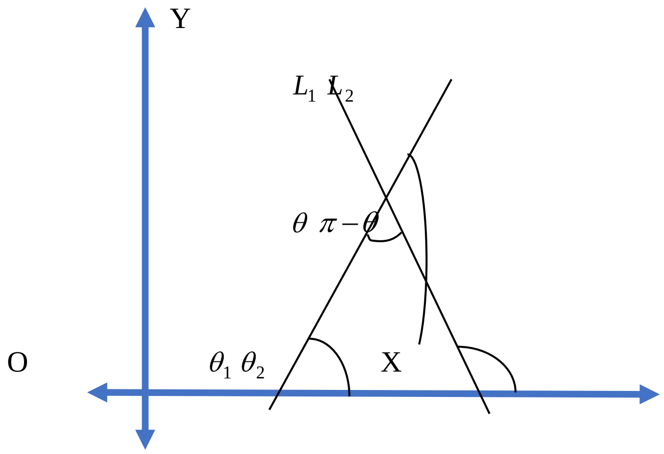


Fig. 3.12

Let $L_1 \,\&\, L_2$ be straight lines and $m_1 \,\&\, m_2$ be their slopes respectively. Also, let $\theta_1 \,\&\, \theta_2$ be the angles which $L_1 \,\&\, L_2$ make with positive X-axis respectively.

Therefore $m_1 = \tan(\theta_1) \,\&\, m_2 = \tan(\theta_2)$. Let θ be the acute angle between lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \qquad or \qquad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right|$$

Example 46. Find the acute angle between the lines whose slopes are 1 and 0.

Sol. Given that slopes of lines are 1 and 0.

Let $m_1 = 1$ and $m_2 = 0$.

Also let θ be the acute angle between lines. Therefore, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{1-0}{1+(1)(0)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{1}{1+0}\right| = 1$$

$$\Rightarrow \qquad \tan\theta = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \qquad \theta = \frac{\pi}{4}$$

which is the required acute angle.

Example 47. Find the acute angle between the lines whose slopes are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Sol. Given that slopes of lines are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Let $m_1 = 2+\sqrt{3}$ and $m_2 = 2-\sqrt{3}$.

Also let θ be the acute angle between lines.

Therefore, $\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\left(2+\sqrt{3}\right)-\left(2-\sqrt{3}\right)}{1+\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{2+\sqrt{3}-2+\sqrt{3}}{1+(4-3)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{2\sqrt{3}}{2}\right| = \sqrt{3}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \qquad \theta = \frac{\pi}{3}$$

which is the required acute angle.

Example 48. Find the obtuse angle between the lines whose slopes are $\sqrt{3}$ and $\dfrac{1}{\sqrt{3}}$.

Sol. Given that slopes of lines are $\sqrt{3}$ and $\dfrac{1}{\sqrt{3}}$. Let $m_1 = \sqrt{3}$ and $m_2 = \dfrac{1}{\sqrt{3}}$.

Also let θ be the acute angle between lines. Therefore, $\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{1+\left(\sqrt{3}\right)\left(\frac{1}{\sqrt{3}}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{\frac{3-1}{\sqrt{3}}}{1+1} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{2}{2\sqrt{3}} \right|$$

$$\Rightarrow \qquad \tan\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(30^\circ\right)$$

$$\Rightarrow \qquad \theta = 30^\circ$$

Therefore, $180^\circ - \theta$ is the obtuse angle between the lines.

i.e. $180^\circ - 30^\circ = 150^\circ$ is the obtuse angle between the lines.

Example 49. Find the angle between the lines whose slopes are -3 and 5.

Sol. Given that slopes of lines are -3 and 5. Let $m_1 = -3$ and $m_2 = 5$.

Also let θ be the angle between lines. Therefore, $\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$$\Rightarrow \qquad \tan\theta = \left| \frac{-3-5}{1+(-3)(5)} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{-8}{1-15} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{-8}{-14} \right|$$

$$\Rightarrow \qquad \tan\theta = \frac{4}{7}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{4}{7}\right)$$

which is the required angle.

Example 50. Find the angle between the lines joining the points $(0,0)$, $(2,3)$ and $(2,-2)$, $(3,5)$.

Sol. Let m_1 be the slope of the line joining $(0,0)$ and $(2,3)$ and m_2 be the slope of the line joining $(2,-2)$ and $(3,5)$. Then

$$\Rightarrow \qquad m_1 = \frac{3-0}{2-0} = \frac{3}{2}$$

and $m_2 = \frac{5-(-2)}{3-2} = \frac{7}{1} = 7$.

Also let θ be the angle between lines.

Therefore, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{3}{2} - 7}{1 + \left(\frac{3}{2}\right)(7)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{3-14}{2}}{\frac{2+21}{2}}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{\frac{-11}{2}}{\frac{23}{2}}\right|$$

$$\Rightarrow \qquad \tan\theta = \frac{11}{23}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{11}{23}\right)$$

which is the required angle.

Example 51. Find the angle between the lines joining the points $(6,-5)$, $(-2,1)$ and $(0,3)$, $(-8,6)$.

Sol. Let m_1 be the slope of the line joining $(6,-5)$ and $(-2,1)$ and m_2 be the slope of the line joining $(0,3)$ and $(-8,6)$. Then

$$\Rightarrow \qquad m_1 = \frac{1-(-5)}{-2-6} = -\frac{6}{8} = -\frac{3}{4}$$

and $m_2 = \dfrac{6-3}{-8-0} = -\dfrac{3}{8}$.

Also let θ be the angle between lines. Therefore, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{3}{4} - \left(-\frac{3}{8}\right)}{1 + \left(-\frac{3}{4}\right)\left(-\frac{3}{8}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{3}{4} + \frac{3}{8}}{1 + \frac{9}{32}}\right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{\frac{-6+3}{8}}{\frac{32+9}{32}} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{\frac{-3}{8}}{\frac{41}{32}} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{-3}{8} \times \frac{32}{41} \right|$$

$$\Rightarrow \qquad \tan\theta = \frac{12}{41}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{12}{41}\right)$$

which is the required angle.

Example 52. Find the angle between the pair of straight lines

$(-2+\sqrt{3})x + y + 9 = 0$ and $(2+\sqrt{3})x - y + 20 = 0$.

Sol. Given equations of lines

$$(-2+\sqrt{3})x + y + 9 = 0 \qquad (1)$$

and $$(2+\sqrt{3})x - y + 20 = 0 \qquad (2)$$

Let m_1 be the slope of the line (1) and m_2 be the slope of the line (2).

$$\Rightarrow \qquad m_1 = -\frac{-2+\sqrt{3}}{1} = 2-\sqrt{3}$$

and $m_2 = -\dfrac{2+\sqrt{3}}{-1} = 2+\sqrt{3}$.

Also let θ be the angle between lines.

Therefore, $\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

$$\Rightarrow \qquad \tan\theta = \left| \frac{(2-\sqrt{3})-(2+\sqrt{3})}{1+(2-\sqrt{3})(2+\sqrt{3})} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{2-\sqrt{3}-2-\sqrt{3}}{1+4-3} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| \frac{-2\sqrt{3}}{2} \right|$$

$$\Rightarrow \qquad \tan\theta = \left| -\sqrt{3} \right|$$

$$\Rightarrow \qquad \tan\theta = \sqrt{3}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \qquad \theta = \frac{\pi}{3}$$

which is the required angle.

Example 53. Find the angle between the pair of straight lines

$x + \sqrt{3}\,y - 8 = 0$ and $x - \sqrt{3}\,y + 2 = 0$.

Sol. Given equations of lines

$$x + \sqrt{3}\,y - 8 = 0 \tag{1}$$

and
$$x - \sqrt{3}\,y + 2 = 0 \tag{2}$$

Let m_1 be the slope of the line (1) and m_2 be the slope of the line (2).

$$\Rightarrow \qquad m_1 = -\frac{1}{\sqrt{3}}$$

and $m_2 = -\dfrac{1}{-\sqrt{3}} = \dfrac{1}{\sqrt{3}}$.

Also let θ be the angle between lines. Therefore, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \left(-\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{2}{\sqrt{3}}}{1 - \frac{1}{3}}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{2}{\sqrt{3}}}{\frac{3-1}{3}}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|\frac{-\frac{2}{\sqrt{3}}}{\frac{2}{3}}\right|$$

$$\Rightarrow \qquad \tan\theta = \left|-\frac{3}{\sqrt{3}}\right|$$

$$\Rightarrow \qquad \tan\theta = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{3\sqrt{3}}{3}$$

$$\Rightarrow \qquad \tan\theta = \sqrt{3}$$

$$\Rightarrow \qquad \tan\theta = \tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \qquad \theta = \frac{\pi}{3}$$

which is the required angle.

Example 54. Find the equations of straight lines making an angle 45° with the line

$6x + 5y - 1 = 0$ and passing through the point $(2, -1)$.

Sol. Given that equations of lines are

$$6x + 5y - 1 = 0 \qquad (1)$$

Let m_1 be the slope of the line (1)

$$\Rightarrow \qquad m_1 = -\frac{6}{5}.$$

Let m_2 be the slope of required line. Therefore, $\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \qquad 1 = \left|\frac{-\frac{6}{5} - m_2}{1 + \left(-\frac{6}{5}\right)(m_2)}\right|$$

$$\Rightarrow \qquad 1 = \left|\frac{\frac{-6-5m_2}{5}}{\frac{5-6m_2}{5}}\right|$$

$$\Rightarrow \qquad 1 = \left|\frac{-6-5m_2}{5-6m_2}\right|$$

$$\Rightarrow \qquad 1 = \pm\left(\frac{-6-5m_2}{5-6m_2}\right)$$

$$\Rightarrow \qquad 5 - 6m_2 = \pm(-6 - 5m_2)$$

Taking positive sign, we get

$$5 - 6m_2 = +(-6 - 5m_2)$$

$$\Rightarrow \qquad 5 - 6m_2 = -6 - 5m_2$$

$$\Rightarrow \qquad -m_2 = -11$$

$$\Rightarrow \qquad m_2 = 11$$

So, equation of line passing through $(2, -1)$ with slope 11 is

$$y+1=11(x-2)$$

$\Rightarrow \quad y+1=11x-22$

$\Rightarrow \quad 11x-y-23=0$ (2)

Now taking negative sign, we get

$$5-6m_2=-(-6-5m_2)$$

$\Rightarrow \quad 5-6m_2=6+5m_2$

$\Rightarrow \quad -11m_2=1$

$\Rightarrow \quad m_2=-\dfrac{1}{11}$

So, equation of line passing through $(2,-1)$ with slope $-\frac{1}{11}$ is

$$y+1=-\frac{1}{11}(x-2)$$

$\Rightarrow \quad 11y+11=-x+2$

$\Rightarrow \quad x+11y+9=0$ (3)

(2) and (3) are required equations of straight lines.

EXERCISE – II

1. The slope of Y-axis is:

(a) Infinite (b) 0 (c) 1/2 (d) 1

2. If two lines are intersecting at an angle of 60^0 then, the other angle between these two lines is:
(a) 120^0 (b) 60^0 (c) 90^0 (d) 180^0

3. If the equation of straight line is ax + by + c = 0, then slope of straight line is:

(a) $-\frac{b}{a}$ (b) $-\frac{a}{b}$ (c) $\frac{b}{a}$ (d) c

4. The equation of a straight line passing through (x_1, y_1) and having slope m is:

(a) y - y_1 = m (x - x_1) (b) x - x_1 = m (y - y_1)

(c) y - y_1 = - m (x - x1) (d) None of these

5. Find the straight line which passes through the following pairs of points.

(i) (-11, -5), (-3, 10) (ii) (0, 0), (10, -12)

6. Find the equation of straight line which makes an angle 60^0 with x-axis and cuts on intercepts 5 on y-axis above the x-axis.

7. Find the equation of straight line which passes through (2, 3) and makes equal intercepts in sign and magnitude on axes.

8. Find the equation of straight line passes through (2, 4) and sum of whose intercepts on axes is 15.

9. Find the equation of straight line such that the length of perpendicular from the origin to the straight line is 10 and the inclination of this perpendicular to the x-axis is 60^0.

10. Find the angle between the lines joining the points (0, 0), (4, 6) and (1, -1), (6, 10)

11. Find the angle between the pair of straight lines $\left(-4+\sqrt{3}\right)$ x + y + 9 = 0 and $\left(4+\sqrt{3}\right)$ x - y + 10 = 0.

12. Find the equation of straight line making an angle 60^0 with the line 6x + 5y -1 = 0 and passing through the point (1, -1).

ANSWERS

1. (a) 2. (a) 3. (b) 4. (a) 5. (i) 15x-8y+125=0 (ii) 6x+15y = 0 6. $y=\sqrt{3}x+5$

7. .x+y=5 8. $\frac{x}{10}+\frac{y}{5}=1$; $\frac{x}{3}+\frac{y}{12}=1$ 9. $\frac{x}{2}+\frac{\sqrt{3}}{2}y=10$ 10. $\theta=\tan^{-1}(7/43)$

11. $\theta=\tan^{-1}\left(-\sqrt{3}/7\right)$ 12. $y+1=\left(\frac{5\sqrt{3}-6}{6\sqrt{3}+5}\right)(x-2)$; $y+1=\left(\frac{5\sqrt{3}+6}{6\sqrt{3}-5}\right)(x-2)$

3.3 CIRCLE

Circle: Circle is the locus of a point which moves in a plane such that its distance from a fixed point always remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

In figure 3.13, $C(h,k)$ be the centre of the circle,

$r$be the radius of the circle and $P(x,y)$ be the

moving point on the circumference of the circle.

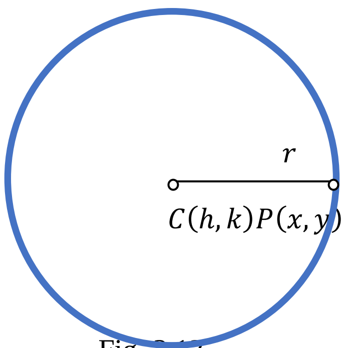


Fig. 3.13

Standard form of Equation of Circle: Let $C(h,k)$ be the centre of the circle, r be the radius of the circle and $P(x,y)$ be any point on the circle, then equation of circle is

$$(x-h)^2+(y-k)^2=r^2 \qquad (1)$$

which is known as standard form of equation of circle. This is also known as central form of equation of circle.

<u>Some Particular Cases</u>:

Let $C(h,k)$ be the centre of the circle, r be the radius of the circle and $P(x,y)$ be any point on the circle:

(i) When the centre of the circle coincides with the origin i.e. $h = k = 0$: (see Fig. 3.14)

Thus equation (1) becomes:

$$\Rightarrow \quad (x-0)^2+(y-0)^2=r^2$$

$$\Rightarrow \quad x^2+y^2=r^2$$

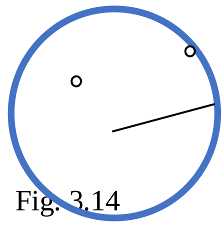


Fig. 3.14

(ii) When the circle passes through the origin: (see figure 3.15)
Let CR be the perpendicular from the centre on X-axis. Therefore,

$$OR^2+CR^2=OC^2$$

$$\Rightarrow \quad (h-0)^2+(k-0)^2=r^2$$

$$\Rightarrow \quad h^2+k^2=r^2$$

Thus equation (1) becomes:

$$\Rightarrow \quad (x-h)^2+(y-k)^2=h^2+k^2$$

$$\Rightarrow \quad x^2+h^2-2hx+y^2+k^2-2ky=h^2+k^2$$

$$\Rightarrow \quad x^2+y^2-2hx-2ky=0$$

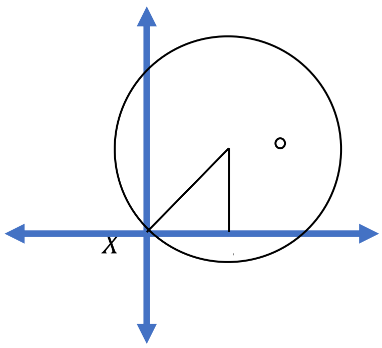


Fig. 3.15

(iii) When the circle passes through the origin and centre lies on the X-axis i.e. $k = 0$: (see figure 3.16)

In this case radius $r = |h|$ Thus equation (1) becomes:

$$\Rightarrow \quad (x-h)^2+(y-0)^2=h^2$$

$$\Rightarrow \quad x^2+h^2-2hx+y^2=h^2$$

$$\Rightarrow \quad x^2+y^2-2hx=0$$

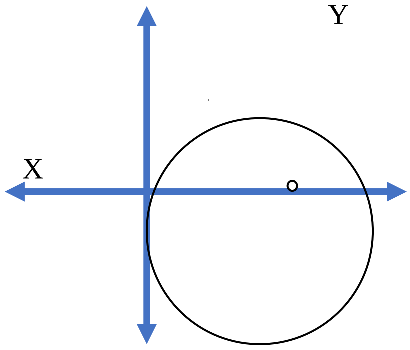
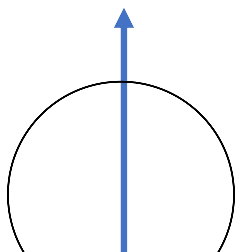


Fig. 3.16

(iv) When the circle passes through the origin and centre lies on the Y-axis i.e. $h = 0$: (see Fig. 3.17)



In this case radius $r = |k|$ Thus equation (1) becomes:

$$\Rightarrow \quad (x-0)^2 + (y-k)^2 = k^2$$

$$\Rightarrow \quad x^2 + y^2 + k^2 - 2ky = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ky = 0$$

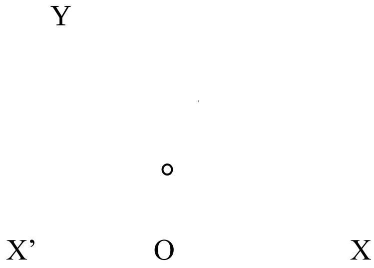


Fig.3.17

(v) When the circle touches the X-axis: (see figure 3.18)

In this case radius $r = |k|$

Thus equation (1) becomes: $\Rightarrow \quad (x-h)^2 + (y-k)^2 = k^2$

$$\Rightarrow \quad x^2 + h^2 - 2hx + y^2 + k^2 - 2ky = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + h^2 = 0$$

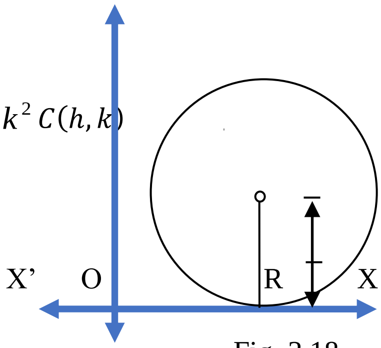


Fig. 3.18

(vi) When the circle touches the Y-axis: (see figure. 3.19)

In this case radius $r = |h|$ Thus equation (1) becomes:

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = h^2$$

$$\Rightarrow \quad x^2 + h^2 - 2hx + y^2 + k^2 - 2ky = h^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + k^2 = 0$$

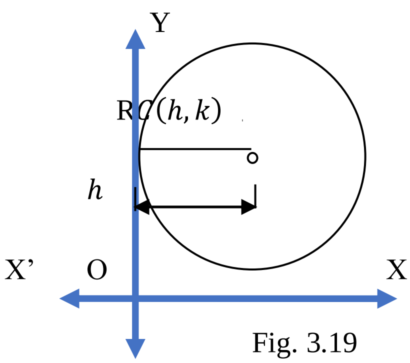


Fig. 3.19

(vii) When the circle touches both the axes: (see figure 3.20)

In this case radius $r = |h| = |k|$ Thus equation (1) becomes:

$$\Rightarrow \quad (x-h)^2 + (y-h)^2 = h^2$$

$$\Rightarrow \quad x^2 + h^2 - 2hx + y^2 + h^2 - 2hy = h^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2hy + h^2 = 0$$

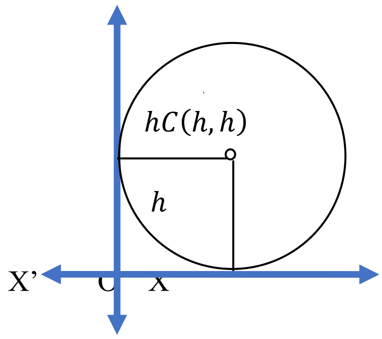


Fig. 3.20

<u>General Equation of Circle</u>:

An equation of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ is known as general equation of circle, where g , f and c are arbitrary constants.

<u>To convert general equation of circle into standard equation</u>:

Let the general equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (2)$$

$$\Rightarrow \quad (x^2 + 2gx) + (y^2 + 2fy) + c = 0$$

$$\Rightarrow \quad (x^2 + 2gx + g^2 - g^2) + (y^2 + 2fy + f^2 - f^2) + c = 0$$

$$\Rightarrow \quad (x+g)^2 - g^2 + (y+f)^2 - f^2 + c = 0$$

$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = f^2 + g^2 - c$$

$$\Rightarrow \quad (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$$

which is the required standard form. Comparing it with $(x-h)^2 + (y-k)^2 = r^2$, we get

$h = -g$, $k = -f$ and $r = \sqrt{g^2 + f^2 - c}$.

Hence, centre of given circle (2) is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$. We observe that the centre of circle (2) is $\left(-\frac{1}{2} \times Coefficient\ of\ x, -\frac{1}{2} \times Coefficient\ of\ y\right)$.

<u>Equation of Circle in Diametric Form</u>:

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of diameter of a circle then the equation of circle is given by

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Example 55 Find the centre and radius of the following circles:

(i) $x^2 + y^2 + 2x + 4y - 4 = 0$ (ii) $x^2 + y^2 - 6x + 10y + 3 = 0$

(iii) $x^2 + y^2 - 3x - 5y - 1 = 0$ (iv) $2\,x^2 + 2\,y^2 + 5x - 6y + 2 = 0$

(v) $3x^2 + 3y^2 - 6x - 15y + 12 = 0$ (vi) $x^2 + y^2 - 12y + 6 = 0$

(vii) $x^2 + y^2 + 10x - 3 = 0$ (viii) $x^2 + y^2 + 7x - 9y = 0$

Sol.

(i) Given that equation of circle is

$x^2 + y^2 + 2x + 4y - 4 = 0$ (1)

Compare (1) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = 2,\ 2f = 4$ and $c = -4$

i.e. $g = 1,\ f = 2$ and $c = -4$.

We know that centre of circle is given by $(-g, -f)$ and radius r is given by $\sqrt{g^2 + f^2 - c}$

Therefore, centre of circle (1) is $(-1, -2)$ and radius r of circle (1) is

$$r = \sqrt{1^2 + 2^2 - (-4)}$$

$$\Rightarrow \quad r = \sqrt{1 + 4 + 4}$$

$$\Rightarrow \quad r = \sqrt{9} = 3$$

(ii) Given that equation of circle is

$x^2 + y^2 - 6x + 10y + 3 = 0$ (2)

Compare (2) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = -6,\ 2f = 10$ and $c = 3$ *i.e.* $g = -3,\ f = 5$ and $c = 3$.

We know that centre of circle is given by $(-g, -f)$ and radius r is given by $\sqrt{g^2 + f^2 - c}$. Therefore, centre of circle (2) is $(3, -5)$ and radius r of circle (2) is

$$r = \sqrt{(-3)^2 + 5^2 - 3}$$

$$\Rightarrow \quad r = \sqrt{9 + 25 - 3}$$

$$\Rightarrow \quad r = \sqrt{31}$$

(iii) Given that equation of circle is

$x^2 + y^2 - 3x - 5y - 1 = 0$ (3)

Compare (3) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = -3,\ 2f = -5$ and $c = -1$

i.e. $g = -\frac{3}{2},\ f = -\frac{5}{2}$ and $c = -1$.

We know that centre of circle is given by $(-g, -f)$ and radius r is given by $\sqrt{g^2 + f^2 - c}$

Therefore, centre of circle (3) is $\left(\frac{3}{2}, \frac{5}{2}\right)$ and radius r of circle (3) is

$$r=\sqrt{\left(-\frac{3}{2}\right)^2+\left(-\frac{5}{2}\right)^2-(-1)}$$

$$\Rightarrow \quad r=\sqrt{\frac{9}{4}+\frac{25}{4}+1}$$

$$\Rightarrow \quad r=\sqrt{\frac{9+25+4}{4}}=\sqrt{\frac{38}{4}}$$

$$\Rightarrow \quad r=\sqrt{\frac{19}{2}}$$

(iv) Given that equation of circle is

$$2\,x^2+2\,y^2+5x-6y+2=0$$

Diving this equation by 2, we get

$x^2+\ y^2+\frac{5}{2}x-3y+1=0$ (4)

Compare (4) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=\frac{5}{2}$, $2f=-3$ and $c=1$

i.e. $g=\frac{5}{4}$, $f=-\frac{3}{2}$ and $c=1$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$

Therefore, centre of circle (4) is $\left(-\frac{5}{4},\frac{3}{2}\right)$ and radius r of circle (4) is

$$r=\sqrt{\left(\frac{5}{4}\right)^2+\left(-\frac{3}{2}\right)^2-1}$$

$$\Rightarrow \quad r=\sqrt{\frac{25}{16}+\frac{9}{4}-1}$$

$$\Rightarrow \quad r=\sqrt{\frac{25+36-16}{16}}=\sqrt{\frac{45}{16}}$$

$$\Rightarrow \quad r=\frac{3\sqrt{5}}{4}$$

(v) Given that equation of circle is

$$3x^2+3y^2-6x-15y+12=0$$

Diving this equation by 3, we get

$x^2+\ y^2-2x-5y+4=0$ (5)

Compare (5) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = -2$, $2f = -5$ and $c = 4$

i.e. $g = -1$, $f = -\frac{5}{2}$ and $c = 4$.

We know that centre of circle is given by $(-g, -f)$ and radius r is given by $\sqrt{g^2 + f^2 - c}$

Therefore, centre of circle (5) is $\left(1, \frac{5}{2}\right)$ and radius r of circle (5) is

$$r = \sqrt{(-1)^2 + \left(-\frac{5}{2}\right)^2 - 4}$$

$$\Rightarrow \qquad r = \sqrt{1 + \frac{25}{4} - 4}$$

$$\Rightarrow \qquad r = \sqrt{\frac{4 + 25 - 16}{4}} = \sqrt{\frac{13}{4}}$$

$$\Rightarrow \qquad r = \frac{\sqrt{13}}{2}$$

(vi) Given that equation of circle is

$x^2 + y^2 - 12y + 6 = 0$ (6)

Compare (6) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = 0$, $2f = -12$ and $c = 6$

i.e. $g = 0$, $f = -6$ and $c = 6$. We know that centre of circle is given by $(-g, -f)$

and radius r is given by $\sqrt{g^2 + f^2 - c}$.

Therefore, centre of circle (6) is $(0, 6)$ and radius r of circle (6) is

$$r = \sqrt{(0)^2 + (-6)^2 - 6}$$

$$\Rightarrow \qquad r = \sqrt{0 + 36 - 6}$$

$$\Rightarrow \qquad r = \sqrt{30}$$

(vii) Given that equation of circle is

$x^2 + y^2 + 10x - 3 = 0$ (7)

Compare (7) with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = 10$, $2f = 0$ and $c = -3$

i.e. $g=5$, $f=0$ and $c=-3$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$
Therefore, centre of circle (7) is $(-5,0)$ and radius r of circle (7) is

$$r=\sqrt{(5)^2+(0)^2-(-3)}$$

$$\Rightarrow \quad r=\sqrt{25+0+3}=\sqrt{28}$$

$$\Rightarrow \quad r=2\sqrt{7}$$

(viii) Given that equation of circle is
$x^2+y^2+7x-9y=0$ (8)
Compare (8) with $x^2+y^2+2gx+2fy+c=0$, we get

$2g=7$, $2f=-9$ and $c=0$
i.e. $g=\frac{7}{2}$, $f=-\frac{9}{2}$ and $c=0$.

We know that centre of circle is given by $(-g,-f)$ and radius r is given by $\sqrt{g^2+f^2-c}$
Therefore, centre of circle (8) is $\left(-\frac{7}{2},\frac{9}{2}\right)$ and radius r of circle (8) is

$$r=\sqrt{\left(\frac{7}{2}\right)^2+\left(-\frac{9}{2}\right)^2-0}$$

$$\Rightarrow \quad r=\sqrt{\frac{49}{4}+\frac{81}{4}}$$

$$\Rightarrow \quad r=\sqrt{\frac{49+81}{4}}=\sqrt{\frac{130}{4}}$$

$$\Rightarrow \quad r=\sqrt{\frac{65}{2}}$$

Example 56. Find the equations of circles if their centres and radii are as follow:

(i) $(0,0)$, 2 (ii) $(2,0)$, 5 (iii) $(0,-3)$, 3

(iv) $(8,-4)$, 1 (v) $(3,6)$, 6 (vi) $(-2,-5)$, 10

Sol.

(i) Given that centre of circle is $(0,0)$ and radius is 2 i.e. $h=0$, $k=0$ and $r=2$.
We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \quad (x-0)^2+(y-0)^2=2^2$$

$$\Rightarrow \quad x^2+y^2=4$$

$$\Rightarrow \qquad x^2+y^2-4=0$$

which is the required equation of circle.

(ii) Given that centre of circle is $(2,0)$ and radius is 5 i.e. $h=2$, $k=0$ and $r=5$.
We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \qquad (x-2)^2+(y-0)^2=5^2$$

$$\Rightarrow \qquad x^2+4-4x+y^2=25$$

$$\Rightarrow \qquad x^2+y^2-4x-21=0$$

which is the required equation of circle.

(iii) Given that centre of circle is $(0,-3)$ and radius is 3 i.e. $h=0$, $k=-3$ and $r=3$.
We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \qquad (x-0)^2+(y-(-3))^2=3^2$$

$$\Rightarrow \qquad x^2+(y+3)^2=3^2$$

$$\Rightarrow \qquad x^2+y^2+9+6y=9$$

$$\Rightarrow \qquad x^2+y^2+6y=0$$

which is the required equation of circle.

(iv) Given that centre of circle is $(8,-4)$ and radius is 1 i.e. $h=8$, $k=-4$ and $r=1$.
We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow \qquad (x-8)^2+(y-(-4))^2=1^2$$

$$\Rightarrow \qquad (x-8)^2+(y+4)^2=1^2$$

$$\Rightarrow \qquad x^2+64-16x+y^2+16+8y=1$$

$$\Rightarrow \qquad x^2+y^2-16x+8y+79=0$$

which is the required equation of circle.

(v) Given that centre of circle is $(3,6)$ and radius is 6 i.e. $h=3$, $k=6$ and $r=6$.
We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-3)^2+(y-6)^2=6^2$

$\Rightarrow \quad x^2+9-6x+y^2+36-12y=36$

$\Rightarrow \quad x^2+y^2-6x-12y+9=0$

which is the required equation of circle.

(vi) Given that centre of circle is $(-2,-5)$ and radius is 10 i.e. $h=-2$, $k=-5$ and $r=10$. We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-(-2))^2+(y-(-5))^2=10^2$

$\Rightarrow \quad (x+2)^2+(y+5)^2=100$

$\Rightarrow \quad x^2+4+4x+y^2+25+10y=100$

$\Rightarrow \quad x^2+y^2+4x+10y-71=0$

which is the required equation of circle.

Example 57. Find the equation of circle whose centre coincides with origin and radius is 4.

Sol. Given that centre of circle coincides with origin i.e. $h=0$, $k=0$ and radius $r=4$.

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-0)^2+(y-0)^2=4^2$

$\Rightarrow \quad x^2+y^2=16$

$\Rightarrow \quad x^2+y^2-16=0$

which is the required equation of circle.

Example 58. Find the equation of circle whose centre is $(-5,4)$ and passes through the origin.

Sol. Given that centre of circle is $(-5,4)$ i.e. $h=-5$, $k=4$.

Also the circle passes through origin.

Therefore radius is given by

$$r=\sqrt{(-5-0)^2+(4-0)^2}$$

$\Rightarrow \quad r=\sqrt{25+16}=\sqrt{41}$

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x+5)^2+(y-4)^2=\left(\sqrt{41}\right)^2$

$\Rightarrow \quad x^2+25+10x+y^2+16-8y=41$

$\Rightarrow \quad x^2+y^2+10x-8y=0$

which is the required equation of circle.

Example 59. Find the equation of circle with radius 4 whose centre lies on X-axis and passes through the point $(2,-4)$.

Sol. Given that centre of circle lies on X-axis. Let the centre is $(h,0)$ i.e. $k=0$. Also, given that radius $r=4$. We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-h)^2+(y-0)^2=(4)^2$

$\Rightarrow \quad x^2+h^2-2hx+y^2=16$

$\Rightarrow \quad x^2+y^2-2hx+h^2-16=0 \qquad \ldots\ldots \qquad (1)$

Also the circle passes through $(2,-4)$. Put $x=2$ and $y=-4$ in (1), we get

$$2^2+(-4)^2-2h(2)+h^2-16=0$$

$\Rightarrow \quad 4+16-4h+h^2-16=0$

$\Rightarrow \quad h^2-4h+4=0$

$\Rightarrow \quad h^2-2h-2h+4=0$

$\Rightarrow \quad h(h-2)-2(h-2)=0$

$\Rightarrow \quad (h-2)(h-2)=0$

$\Rightarrow \quad h-2=0$

$\Rightarrow \quad h=2$

Put this value in (1), we get

$$x^2+y^2-2(2)x+2^2-16=0$$

$\Rightarrow \quad x^2+y^2-4x+4-16=0$

$\Rightarrow \quad x^2+y^2-4x-12=0$

which is the required equation of circle.

Example 60. Find the equation of circle with radius 3 whose centre lies on Y-axis and passes through the point $(-3,1)$.

Sol. Given that centre of circle lies on Y-axis. Let the centre is $(0,k)$ i.e. $h=0$.

Also, given that radius $r=3$ We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-0)^2+(y-k)^2=(3)^2$

$\Rightarrow \quad x^2+y^2+k^2-2ky=9$

$\Rightarrow \quad x^2+y^2+k^2-2ky-9=0 \qquad \ldots . (1)$

Also the circle passes through $(-3,1)$. Put $x=-3$ and $y=1$ in (1), we get

$$(-3)^2+(1)^2+k^2-2k(1)-9=0$$

$\Rightarrow \quad 9+1+k^2-2k-9=0$

$\Rightarrow \quad k^2-2k+1=0$

$\Rightarrow \quad k^2-k-k+1=0$

$\Rightarrow \quad k(k-1)-1(k-1)=0$

$\Rightarrow \quad (k-1)(k-1)=0$

$\Rightarrow \quad k-1=0$

$\Rightarrow \quad k=1$

Put this value in (1), we get

$$x^2+y^2+1^2-2(1)y-9=0$$

$\Rightarrow \quad x^2+y^2+1-2y-9=0$

$\Rightarrow \quad x^2+y^2-2y-8=0$

which is the required equation of circle.

Example 61. Find the equation of circle which touches the Y-axis with centre $(-3,1)$.

Sol. Given that centre of circle is $(-3,1)$ i.e. $h=-3$ and $y=1$.

Also the circle touches the Y-axis. Therefore $r=|h|=|-3|=3$

We know that the equation of circle, when centre and radius is given, is

$$(x-h)^2+(y-k)^2=r^2$$

$\Rightarrow \quad (x-(-3))^2+(y-1)^2=(3)^2$

$\Rightarrow \quad (x+3)^2+(y-1)^2=9$

$\Rightarrow \quad x^2+9+6x+y^2+1-2y-9=0$

$\Rightarrow \quad x^2+y^2+6x-2y+1=0$

which is the required equation of circle.

Example 62. Find the equation of circles if end points of their diameters are as follow:

(i) $(1,5)$ and $(3,6)$

(ii) $(1,0)$ and $(-2,-5)$

(iii) $(0,0)$ and $(8,-6)$

(iv) $(-3,2)$ and $(-7,9)$

Sol.

(i) Given that end points of diameter of circle are $(1,5)$ and $(3,6)$.
Comparing these points with (x_1, y_1) and $(x_2\,, y_2)$ respectively, we get
$x_1 = 1$, $y_1 = 5$, $x_2 = 3$ and $y_2 = 6$. We know that the equation of circle in diametric form is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$$\Rightarrow \quad (x-1)(x-3)+(y-5)(y-6)=0$$
$$\Rightarrow \quad x^2-3x-x+3+y^2-6y-5y+30=0$$
$$\Rightarrow \quad x^2+y^2-4x-11y+33=0$$

which is the required equation of circle.

(ii) Given that end points of diameter of circle are $(1,0)$ and $(-2,-5)$.
Comparing these points with (x_1, y_1) and $(x_2\,, y_2)$ respectively, we get
$x_1 = 1$, $y_1 = 0$, $x_2 = -2$ and $y_2 = -5$.
We know that the equation of circle in diametric form is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$
$$\Rightarrow \quad (x-1)(x-(-2))+(y-0)(y-(-5))=0$$
$$\Rightarrow \quad (x-1)(x+2)+y(y+5)=0$$
$$\Rightarrow \quad x^2+2x-x-2+y^2+5y=0$$
$$\Rightarrow \quad x^2+y^2+x+5y-2=0$$

which is the required equation of circle.

(iii) Given that end points of diameter of circle are $(0,0)$ and $(8,-6)$.
Comparing these points with (x_1, y_1) and $(x_2\,, y_2)$ respectively, we get
$x_1 = 0$, $y_1 = 0$, $x_2 = 8$ and $y_2 = -6$. We know that the equation of circle in diametric form is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$$\Rightarrow \quad (x-0)(x-8)+(y-0)(y-(-6))=0$$
$$\Rightarrow \quad x(x-8)+y(y+6)=0$$
$$\Rightarrow \quad x^2-8x+y^2+6y=0$$
$$\Rightarrow \quad x^2+y^2-8x+6y=0$$

which is the required equation of circle.

(iv) Given that end points of diameter of circle are $(-3,2)$ and $(-7,9)$.
Comparing these points with (x_1, y_1) and $(x_2\,, y_2)$ respectively, we get
$x_1 = -3$, $y_1 = 2$, $x_2 = -7$ and $y_2 = 9$.
We know that the equation of circle in diametric form is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$
$$\Rightarrow \quad (x-(-3))(x-(-7))+(y-2)(y-9)=0$$
$$\Rightarrow \quad (x+3)(x+7)+(y-2)(y-9)=0$$

$$\Rightarrow \quad x^2 + 7x + 3x + 21 + y^2 - 9y - 2y + 18 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 10x - 11y + 39 = 0$$

which is the required equation of circle.

EXERCISE - III

1. Equation of circle with centre at (2,0) and radius 7 is:

(a) $x^2 + 4 - 4x + y^2 = 14$ (b) $x^2 + 4 - 4x + y^2 = 49$

(c) $x^2 - 4 + 4x + y^2 = 49$ (d) None of these

2. Equation of a circle whose centre is origin and radius v is:

(a) $x^2 + y^2 + 2gn + 2fy + c = 0$ (b) $x^2 + y^2 = v^2$

(c) $x^2 - 2vx + y^2 = v^2$ (d) $x^2 + y^2 = 0$

3. Equation of circle in diametric form is:

(a) $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ (b) $(x - x_1)(y - y_1) + (x - x_2)(y - y_2) = 0$

(c) $(x - y_1)(x - x_2) + (y - x_1)(y - x_2) = 0$ (d) $(x - y)(x - y) + (y_1 - x_1)(y_2 - x_2) = 0$

4. Find the centre and radius of the following:

(i) $9x^2 + 9y^2 - 12x - 30y + 24 = 0$

(ii) $x^2 + y^2 - 6y - 24 = 0$

(iii) $x^2 + y^2 + 20x - 5 = 0$

5. Find the equation of circles of their centre and radii are as follows:

(i) (8, 8) , 2 (ii) (6, 3) , 6

6. Find the equation of the circle with radius 3 whose centre lies on Y-axis and passes through the point (2, -2).

7. Find the equations of circles if end points of their diameters are as follows:

(i) (2, 6) and (3, 16) (ii) (-1, -1) and (4, 5)

ANSWERS

1. (b) 2. (b) 3.(a)

4. (i) $(2/3, 5/3); \sqrt{5/9}$ (ii) $(0,3); \sqrt{33}$ (iii) $(-10,0); \sqrt{105}$

5. (i) $x^2 + y^2 - 16x - 16y + 124 = 0$ (ii) $x^2 + y^2 - 12x - 6y + 9 = 0$

6. $x^2 + y^2 - 2(-2+\sqrt{5})y - 4\sqrt{5} = 0;$ $x^2 + y^2 + 2(2+\sqrt{5})y + 4\sqrt{5} = 0;$

7. (i) $x^2 + y^2 - 5x - 22y + 102 = 0$ (ii) $x^2 + y^2 - 3x - 4y - 9 = 0$

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